Monday Night Calculus

The Method of Substitution

Exercises

1. Evaluate the indefinite integral.

(a) $\displaystyle\int x^2 \cos x^3 \, dx$

Let $u = x^3 \quad\Rightarrow\quad du = 3x^2\, dx \quad\Rightarrow\quad dx = \dfrac{du}{3x^2}$

$$\int x^2 \cos x^3 \, dx = \int x^2 \cos u \cdot \frac{du}{3x^2} = \frac{1}{3}\int \cos u \, du \qquad \text{Change variables; simplify}$$

$$= \frac{1}{3}\sin u + C = \frac{1}{3}\sin x^3 + C \qquad \text{Basic antidifferentiation rule; final answer in terms of } x$$

(b) $\displaystyle\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx$

Let $u = \cos x \quad\Rightarrow\quad du = -\sin x\, dx \quad\Rightarrow\quad dx = -\dfrac{du}{\sin x}$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx = \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) \qquad \text{Change variables}$$

$$= -\int \frac{1}{u}\, du = -\ln|u| + C \qquad \text{Simplify; basic antidifferentiation rule}$$

$$= -\ln|\cos x| + C \qquad \text{Final answer in terms of } x$$

(c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Let $u = \sqrt{x} = x^{1/2} \quad\Rightarrow\quad du = \dfrac{1}{2}x^{-1/2}\, dx \quad\Rightarrow\quad dx = 2\sqrt{x}\, du$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \int \frac{e^u}{\sqrt{x}} \cdot 2\sqrt{x}\, du = 2\int e^u \, du \qquad \text{Change variables; simplify}$$

$$= 2e^u + C = 2e^{\sqrt{x}} + C \qquad \text{Basic antidifferentiation rule; final answer in terms of } x$$

2. (a) Find $\int 2\tan x \sec^2 x\,dx$ using the substitution $u = \tan x$.

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x\,dx \quad \Rightarrow \quad dx = \frac{du}{\sec^2 x}$$

$$\int 2\tan x \sec^2 x\,dx = \int 2u \sec^2 x \cdot \frac{du}{\sec^2 x} = 2\int u\,du$$

Change variables; simplify

$$= 2\cdot\frac{u^2}{2} + C = \tan^2 x + C$$

Basic antidifferentiation rule; final answer in terms of x

(b) Find $\int 2\tan x \sec^2 x\,dx$ using the substitution $u = \sec x$.

$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\,dx \quad \Rightarrow \quad dx = \frac{du}{\sec x \tan x}$$

$$\int 2\tan x \sec^2 x\,dx = \int 2\tan x \sec x \sec x\,dx$$

Isolate du

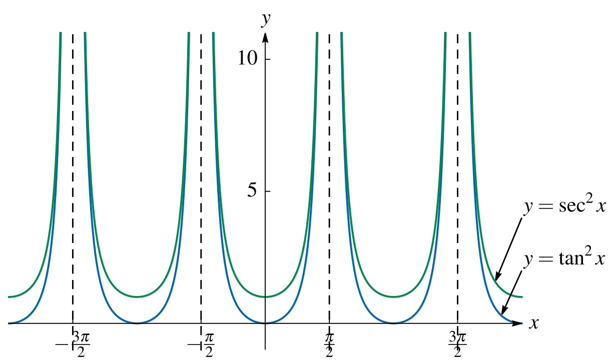
$$= \int 2\tan x \sec x \cdot u \cdot \frac{du}{\sec x \tan x} = 2\int u\,du$$

Change variables; simplify

$$= 2\cdot\frac{u^2}{2} + C = \sec^2 x + C$$

Basic antidifferentiation rule; final answer in terms of x

(c) Graph $y = \tan^2 x$ and $y = \sec^2 x$ in the same viewing window. These functions are clearly different. Explain this observation in connection with parts (a) and (b).



If F and G are two antiderivatives of f, then $F(x) - G(x) = C$.

$\tan^2 x + 1 = \sec^2 x$

3. Find $\displaystyle\int (x^2 + x)\sqrt{2-x}\,dx$

Hint: If $u = 2 - x$, then $x = 2 - u$.

$u = 2 - x \quad \Rightarrow \quad du = -dx \quad \Rightarrow \quad dx = -du; \quad x = 2 - u$

$$\int (x^2 + x)\sqrt{2-x}\,dx = \int (x^2 + x)\sqrt{u} \cdot -du = -\int (x^2 + x)\sqrt{u}\,du \qquad \text{Change variables}$$

$$= -\int ((2-u)^2 + (2-u))\sqrt{u}\,du \qquad x = 2 - u$$

$$= -\int (u^2 - 5u + 6)u^{1/2}\,du \qquad \text{Integrand in terms of } u$$

$$= -\int (u^{5/2} - 5u^{3/2} + 6u^{1/2})\,du \qquad \text{Rewrite integrand}$$

$$= -\left(\frac{2}{7}u^{7/2} - 5\cdot\frac{2}{5}u^{5/2} + 6\cdot\frac{2}{3}u^{3/2}\right) + C \qquad \text{Power Rule}$$

$$= -\left(\frac{2}{7}(2-x)^{7/2} - 2(2-x)^{5/2} + 4(2-x)^{3/2}\right) + C \qquad \text{Final answer in terms of } x$$

4. Suppose $g(x) = f(7-5x)$ and $\int_2^3 g(x)\,dx = c\int_a^b f(x)\,dx$.

Find the values of a, b, and c.

$$u = 7-5x \qquad x = 2:\ u = 7-5(2) = -3$$

$$du = -5\,dx \qquad x = 3:\ u = 7-5(3) = -8$$

$$dx = -\frac{du}{5}$$

$$\int_2^3 g(x)\,dx = \int_2^3 f(7-5x)\,dx \qquad g(x) = f(7-5x)$$

$$= \int_{-3}^{-8} f(u)\cdot -\frac{du}{5} \qquad \text{Change variables}$$

$$= -\frac{1}{5}\int_{-3}^{-8} f(u)\,du \qquad \text{Simplify}$$

$$a = -3,\ \ b = -8,\ \ c = -\frac{1}{5}$$

5. Suppose f is a differentiable function such that $f(1) = 2$ and $f(2) = 3$.

(a) Find $\displaystyle\int_1^2 f'(x)\,dx$

$$\int_1^2 f'(x)\,dx = \Big[f(x)\Big]_1^2 \qquad \text{Antiderivative}$$

$$= f(2) - f(1) = 3 - 2 = 1 \qquad \text{FTC; simplify}$$

(b) Find $\displaystyle\int_1^2 f(x)f'(x)\,dx$

$$u = f(x) \qquad x = 1:\ u = f(1) = 2$$
$$du = f'(x)\,dx \qquad x = 2:\ u = f(2) = 3$$
$$dx = \frac{du}{f(x)}$$

$$\int_1^2 f(x)f'(x)\,dx = \int_2^3 u\,f'(x)\,\frac{du}{f'(x)} = \int_2^3 u\,du \qquad \text{Change variables; simplify}$$

$$= \left[\frac{u^2}{2}\right]_2^3 \qquad \text{Power Rule}$$

$$= \frac{9}{2} - \frac{4}{2} = \frac{5}{2} \qquad \text{FTC; simplify}$$

(c) Find $\displaystyle\int_1^2 \frac{f'(x)}{f(x)}\,dx$

$$u = f(x) \qquad x = 1:\ u = f(1) = 2$$
$$du = f'(x)\,dx \qquad x = 2:\ u = f(2) = 3$$
$$dx = \frac{du}{f(x)}$$

$$\int_1^2 \frac{f'(x)}{f(x)}\,dx = \int_2^3 \frac{f'(x)}{u}\,\frac{du}{f'(x)} = \int_2^3 \frac{du}{u} \qquad \text{Change variables; simplify}$$

$$= \Big[\ln|u|\Big]_2^3 \qquad \text{Basic antidifferentiation rule}$$

$$= \ln 3 - \ln 2 = \ln\frac{3}{2} \qquad \text{FTC; simplify}$$