

## TI in Focus: AP<sup>®</sup> Calculus

2017 AP<sup>®</sup> Calculus Exam: BC-6  
Alternating Series

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## Outline

- (1) Definition
- (2) Alternating Series Test; Geometric Interpretation
- (3) Estimating Sums; Alternating Series Error Bound
- (4) Examples

An **alternating series** is a series whose terms are alternately positive and negative.

Examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{4} - \frac{3}{8} + \frac{4}{16} - \frac{5}{32} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

The  $n$ th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n$$

where  $b_n$  is a positive number.

## Alternating Series Test (AST)

If the alternating series

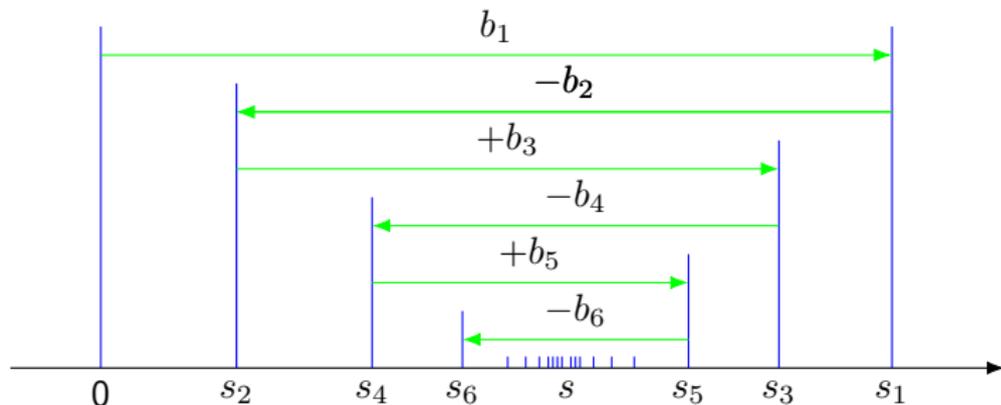
$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \quad b_n > 0$$

satisfies

- (i)  $b_{n+1} \leq b_n$  for all  $n$
- (ii)  $\lim_{n \rightarrow \infty} b_n = 0$

then the series is convergent.

## A geometric look at a convergent alternating series



Compute and plot the partial sums.

Since  $b_n \rightarrow 0$ , and  $b_n$  is decreasing, the successive steps become smaller and smaller.

The even partial sums are increasing. The odd partial sums are decreasing.

It seems reasonable that both converge to some number  $s$ .

## Example 1 Conditions of Worth

Determine whether the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$  converges or diverges.

### Solution

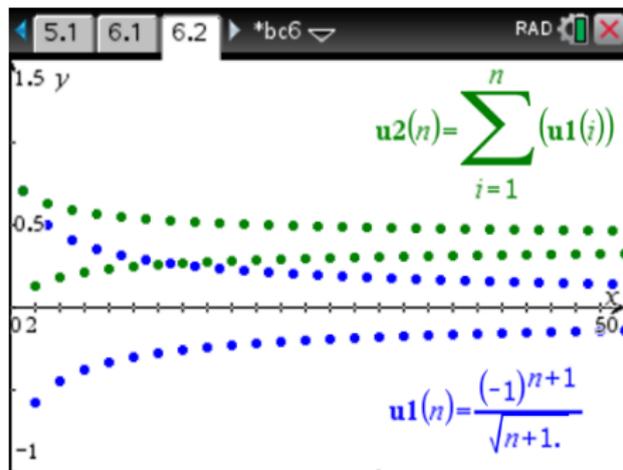
(i)  $b_{n+1} < b_n$  because  $\frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}}$

(ii)  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$

Therefore, the series is convergent by the Alternating Series Test.

## Solution

A graph of the terms  $a_n = \frac{(-1)^{n+1}}{\sqrt{n+1}}$  and the partial sums  $s_n$ .



## Example 2 All Downhill From Here

Determine whether the series  $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$  converges or diverges.

### Solution

(i) It's not obvious that the sequence  $b_n = n e^{-n}$  is decreasing.

Consider the related function  $f(x) = x e^{-x} \implies f'(x) = (1 - x) e^{-x}$

$f$  is decreasing on the interval  $[1, \infty)$ .

This shows that  $f(n + 1) < f(n)$  and  $b_{n+1} < b_n$ .

$$(ii) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{e^n} = \lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$$

Both conditions are satisfied; the series converges by the Alternating Series Test.

## Estimating Sums

- A partial sum  $s_n$  of any convergent series can be used as an approximation to the total sum  $s$ .
- The error in estimation is the remainder  $R_n = s - s_n$
- This theorem provides a bound on the error for a series that satisfies the AST.

## Alternating Series Error Bound Theorem

If  $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ , where  $b_n > 0$ , is the sum of an alternating series that satisfies

- (i)  $b_{n+1} \leq b_n$  for all  $n$
- (ii)  $\lim_{n \rightarrow \infty} b_n = 0$

then the Alternating Series Error Bound is  $|R_n| = |s - s_n| \leq b_{n+1}$ .

**In words:** the alternating series error bound is less than the absolute value of the first omitted term.

error  $< |1\text{st omitted term}|$

### Example 3 Bound and Determined

Consider the alternating series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 4}{5^n(n+1)}$ .

- (a) Show that this series converges.
- (b) Find  $s_6$  and use the Alternating Series Error Bound to show that this value differs from  $s$  by less than  $\frac{1}{150000}$ .

### Solution

- (a) Check the conditions for the AST.

$$(i) \quad b_{n+1} = \frac{4}{5^{n+1}(n+2)} < \frac{4}{5^n(n+1)} = b_n$$

$$(ii) \quad \lim_{n \rightarrow \infty} \frac{4}{5^n(n+1)} = 0$$

Therefore, the series converges by the Alternating Series Test.

**Solution**

$$\begin{aligned} \text{(b)} \quad s_6 &= \frac{2}{5} - \frac{4}{75} + \frac{1}{125} - \frac{4}{3125} + \frac{2}{9375} - \frac{4}{109375} \\ &= \frac{38671}{109375} = 0.353563 \end{aligned}$$

The Alternating Series Error Bound is

$$|s - s_6| \leq b_7 = \frac{4}{5^7(8)} = \frac{4}{625000} = \frac{1}{156250} < \frac{1}{150000}$$

## Solution

6.2 7.1 8.1 \*bc6 ▾ RAD  

$$a(n) := \frac{(-1)^{n-1} \cdot 4}{5^n \cdot (n+1)}$$

Done

seq(a(i), i, 1, 6)

$$\left\{ \frac{2}{5}, \frac{-4}{75}, \frac{1}{125}, \frac{-4}{3125}, \frac{2}{9375}, \frac{-4}{109375} \right\}$$

7.1 8.1 8.2 \*bc6 ▾ RAD  

$$\sum_{i=1}^6 (a(i))$$

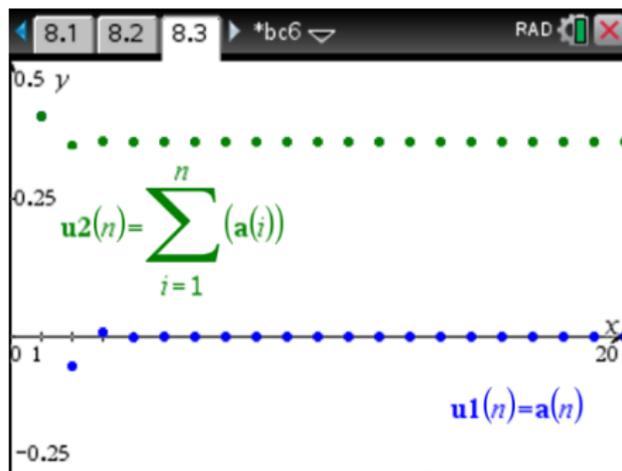
$$\frac{38671}{109375}$$

$$|a(7)|$$

$$\frac{1}{156250}$$

## Solution

Graph of  $\{a_n\}$  and  $\{s_n\}$ .



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