TI in Focus: AP® Calculus

2018 AP® Calculus Exam: BC-5

Technology Solutions and Problem Extensions

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

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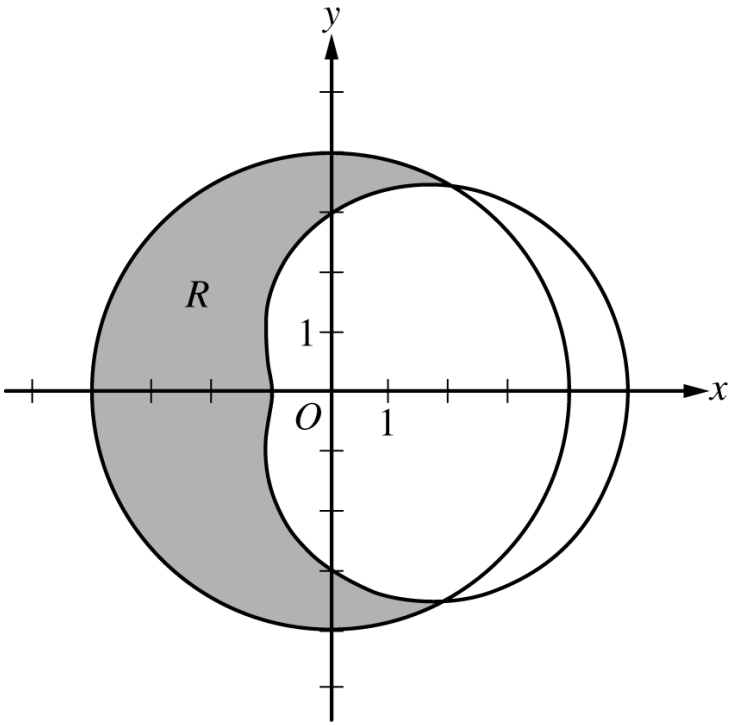
Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions using technology

(4) Problem Extensions



5. The graphs of the polar curves $r = 4$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect at $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

(a) Let R be the shaded region that is inside the graph of $r = 4$ and also outside the graph of $r = 3 + 2\cos\theta$, as shown in the figure above. Write an expression involving an integral for the area of R.

(b) Find the slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$.

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(c) A particle moves along the portion of the curve $r = 3 + 2\cos\theta$ for $0 < \theta < \frac{\pi}{2}$. The particle moves in such a way that the distance between the particle and the origin increases at a constant rate of 3 units per second. Find the rate at which the angle θ changes with respect to time at the instant when the position of the particle corresponds to $\theta = \frac{\pi}{3}$. Indicate units of measure.

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(a) $\text{Area} = \frac{1}{2}\int_{\pi/3}^{5\pi/3}\left(4^2 - (3 + 2\cos\theta)^2\right)d\theta$

$3 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \end{cases}$

(b) $\frac{dr}{d\theta} = -2\sin\theta \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=\pi/2} = -2$

$r\left(\frac{\pi}{2}\right) = 3 + 2\cos\left(\frac{\pi}{2}\right) = 3$

$y = r\sin\theta \Rightarrow \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$x = r\cos\theta \Rightarrow \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{-2\sin\left(\frac{\pi}{2}\right) + 3\cos\left(\frac{\pi}{2}\right)}{-2\cos\left(\frac{\pi}{2}\right) - 3\sin\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

$3 : \begin{cases} 1 : \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \\ \quad \text{or } \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \\ 1 : \frac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ 1 : \text{answer} \end{cases}$

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Part (b) Alternate Solution

$$y = r\sin\theta = (3 + 2\cos\theta)\sin\theta \Rightarrow \frac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$$

$$x = r\cos\theta = (3 + 2\cos\theta)\cos\theta \Rightarrow \frac{dx}{d\theta} = -3\sin\theta - 4\sin\theta\cos\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\pi/2} = \frac{3\cos\left(\frac{\pi}{2}\right) + 2\cos^2\left(\frac{\pi}{2}\right) - 2\sin^2\left(\frac{\pi}{2}\right)}{-3\sin\left(\frac{\pi}{2}\right) - 4\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)} = \frac{2}{3}$$

The slope of the line tangent to the graph of $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{2}$ is $\frac{2}{3}$.

(c) $$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} = -2\sin\theta\cdot\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta}$$

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3\cdot\frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3} \text{ radians per second}$$

$$3 : \begin{cases} 1 : \frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt} \\ 1 : \frac{d\theta}{dt} = \frac{dr}{dt}\cdot\frac{1}{-2\sin\theta} \\ 1 : \text{answer with units} \end{cases}$$

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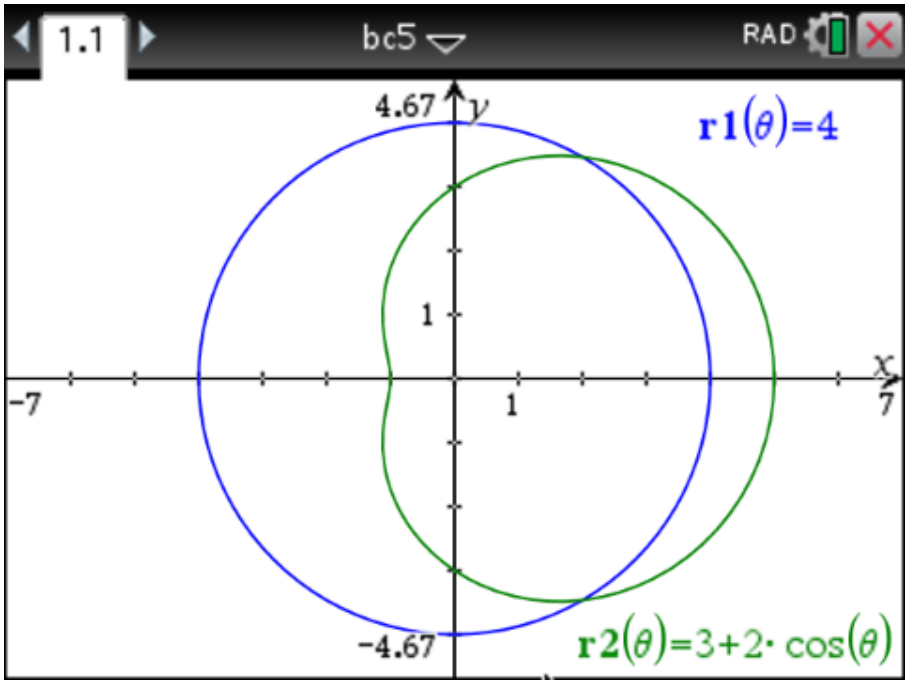
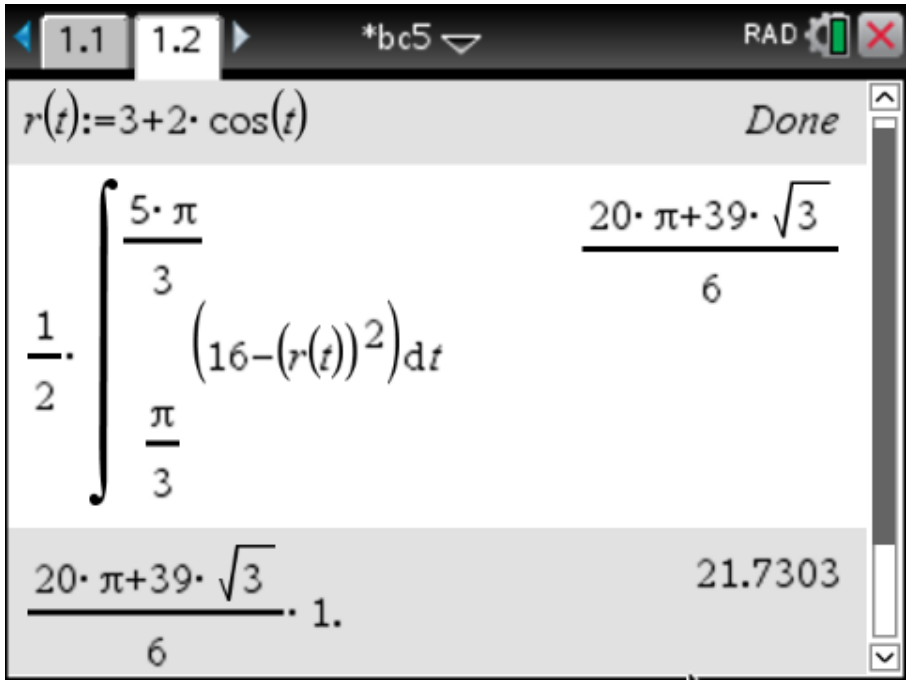
Part (a)

Integral expression for the area of R.

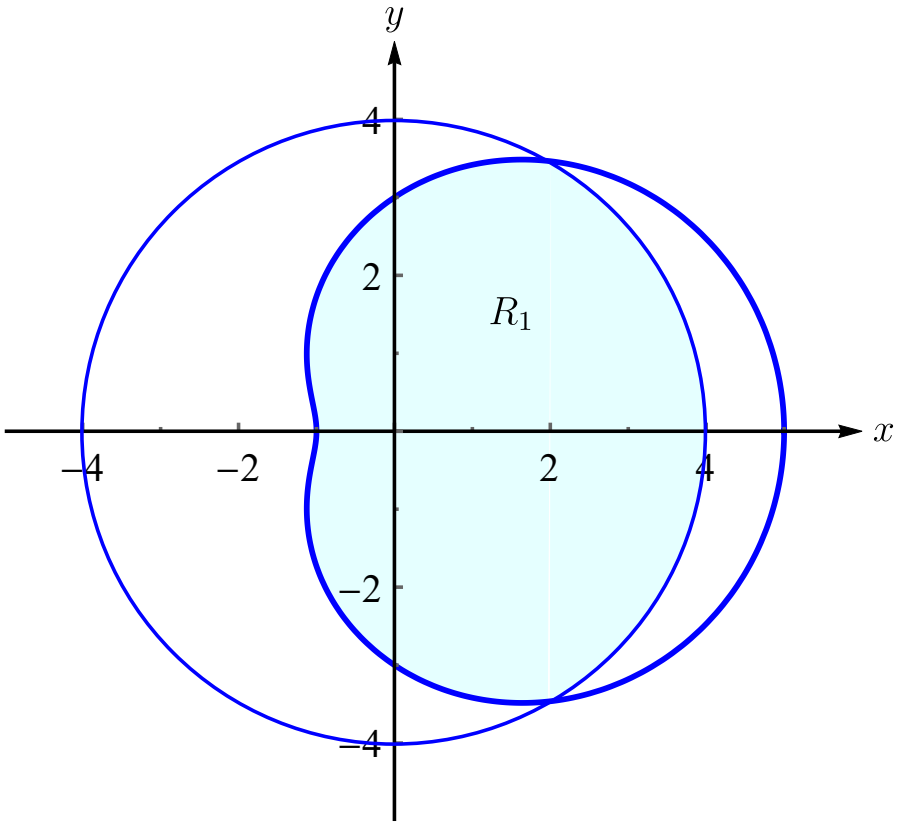
$$\frac{1}{2}\int_{\pi/3}^{5\pi/3}\left[4^2-(3+2\cos\theta)^2\right]\,d\theta$$

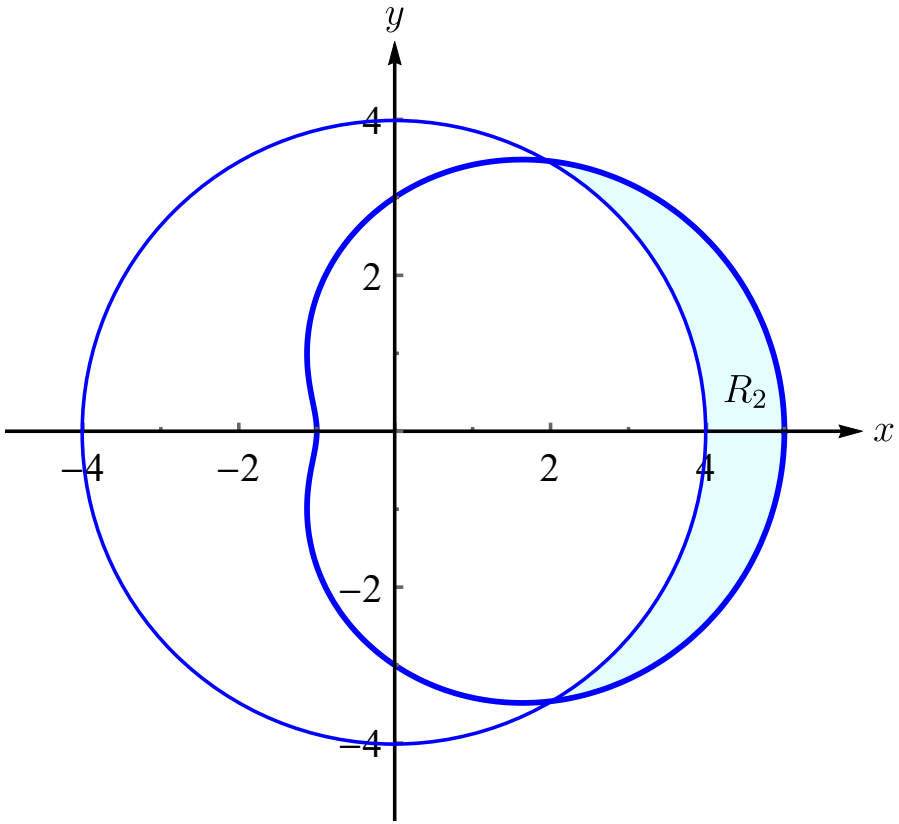




Example 1 Other Regions

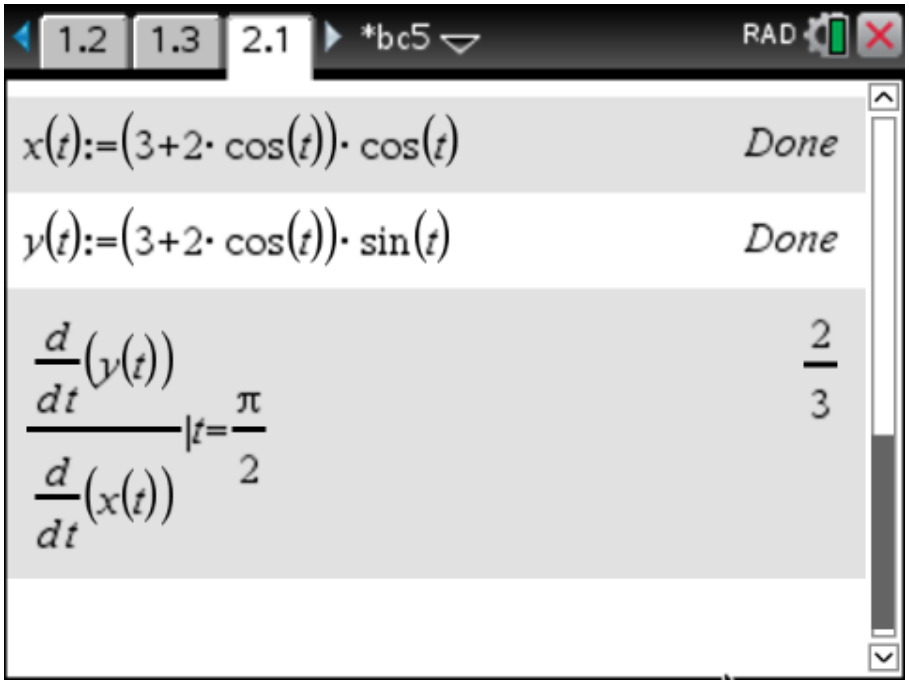
The graphs of $r = 4$ and $r = 3 + 2\cos\theta$ are shown in each figure. The curves intersect at $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{5\pi}{3}$. Find the area of the region R_1 and the area of the region R_2.





Part (b)

Slope of the tangent line to the graph of $r = 3 + 2\cos\theta$ at $\theta = \dfrac{\pi}{2}$

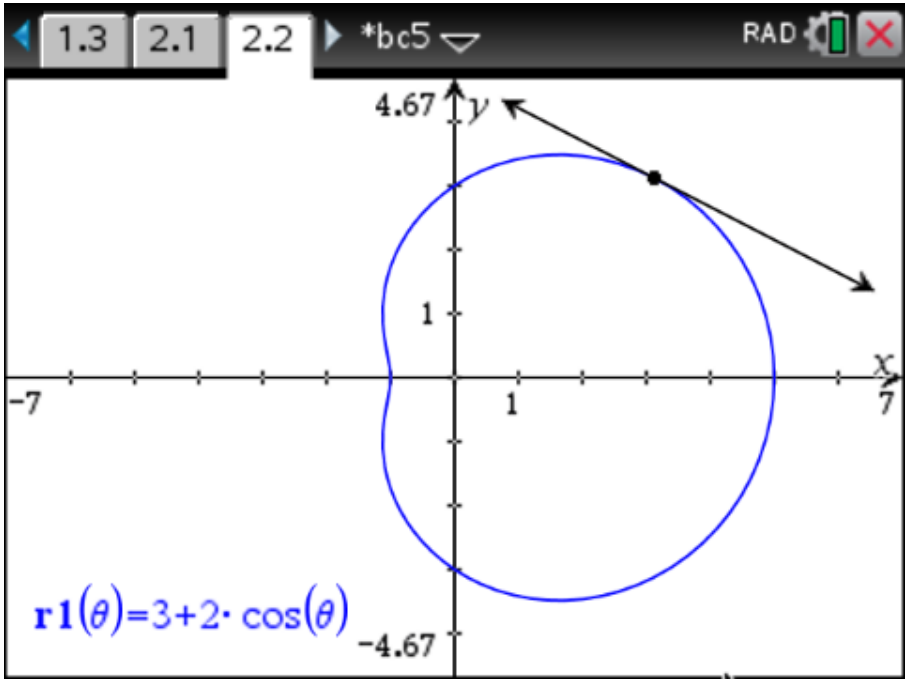


Example 2 Tangent Lines

Consider the curve described by the polar equation $r = 3 + 2\cos\theta$.

(a) Find the slope of the tangent line to the curve when $\theta = \dfrac{\pi}{4}$.

(b) Find the points on the curve where the tangent line is horizontal or vertical.



Part (c) Solution

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt}$$

Chain Rule.

$$= -2\sin\theta \cdot \frac{d\theta}{dt}$$

Compute $\frac{dr}{d\theta}$.

$$\frac{d\theta}{dt} = \frac{dr}{dt} \cdot \frac{1}{-2\sin\theta}$$

Solve for $\frac{d\theta}{dt}$.

$$\left.\frac{d\theta}{dt}\right|_{\theta=\pi/3} = 3 \cdot \frac{1}{-2\sin\left(\frac{\pi}{3}\right)} = \frac{3}{-\sqrt{3}} = -\sqrt{3}$$

Example 3 Related Rates

(a) At noon, ship A is 120 km east of ship B. Ship A is sailing west at 30 km/h and ship B is sailing south at 25 km/h. How fast is the distance between the ships changing at 3:00 pm?

(b) A particle moves along the curve $y = 2\cos\left(\frac{\pi x}{2}\right)$. As the particle passes through the point $\left(\frac{1}{3}, \sqrt{3}\right)$, its x-coordinate decreases at a rate of 5 cm/s. How fast is the distance from the particle to the origin changing at this instant?

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