TI in Focus: AP® Calculus

2018 AP® Calculus Exam: AB-4/BC-4

The Mean Value Theorem

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Outline

(1) Rolle's Theorem

(2) Position Function, Real Roots

(3) The Mean Value Theorem

(4) MVT Illustration

(5) What if $f'(x) = g'(x)$?

Background

(1) The Mean Value Theorem is used to prove many results in mathematics.

(2) The MVT is easy to illustrate, and the result seems reasonable.

(3) Consider a preliminary result first.

Rolle's Theorem

Let f be a function such that

(1) f is continuous on the closed interval $[a, b]$.

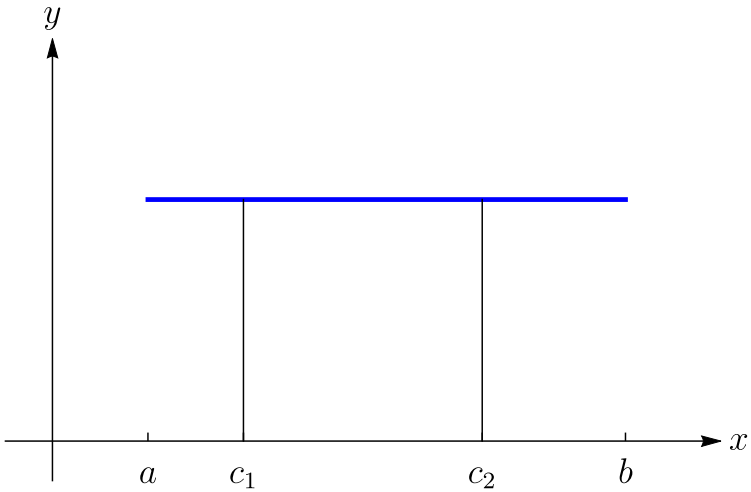
(2) f is differentiable on the open interval (a, b).

(3) $f(a) = f(b)$.

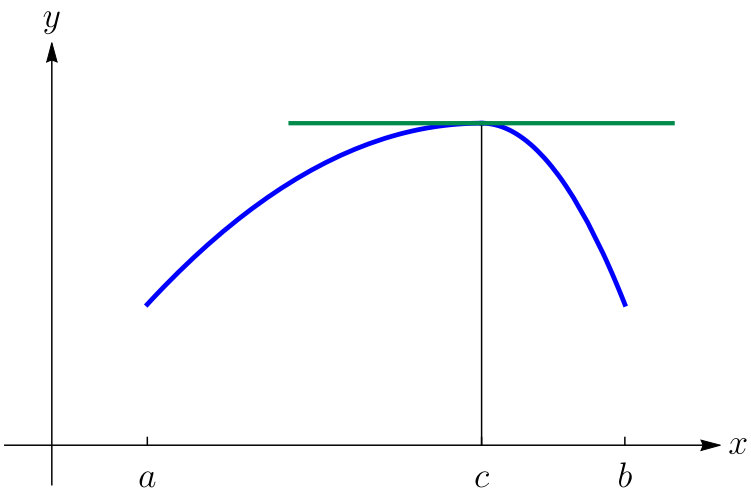
Then there is a number c in (a, b) such that $f'(c) = 0$.

Visualization of Rolle's Theorem

Graphs of functions that satisfy the three conditions of Rolle's Theorem.



$f'(c_1) = f'(c_2) = 0$



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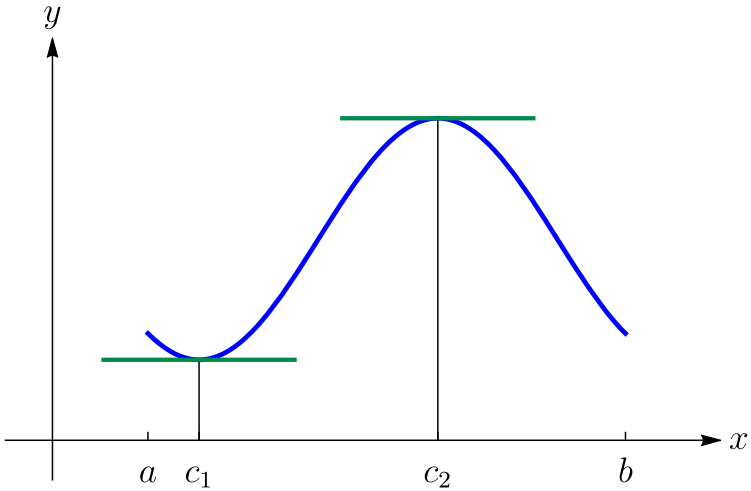

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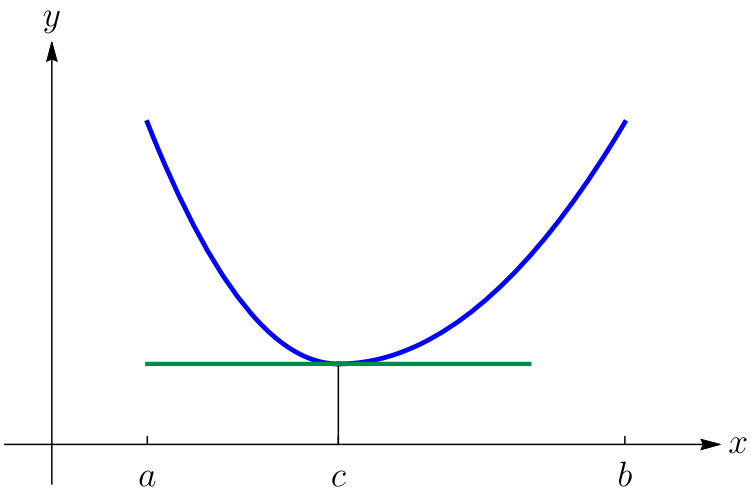


$f'(c) = 0$

Example 1 Rolle's Theorem and a Polynomial

Let $f(x) = -x^3 + 12x^2 - 39x + 40$, $1 \leq x \leq 7$. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

Solution

f is a polynomial: f is continuous on $[1, 7]$; f is differentiable on $(1, 7)$.

$f(1) = -(1)^3 + 12(1)^2 - 39(1) + 40 = 12$

$f(7) = -(7)^3 + 12(7)^2 - 39(7) + 40 = 12$

f satisfies the conditions of Rolle's Theorem.

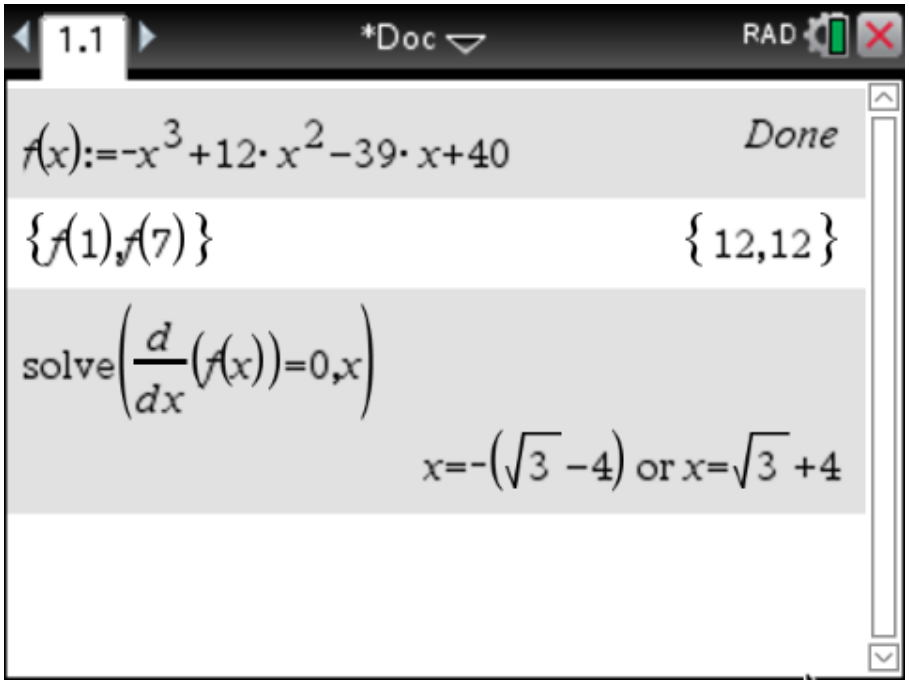
$f'(x) = -3x^2 + 24x - 39 = 0$

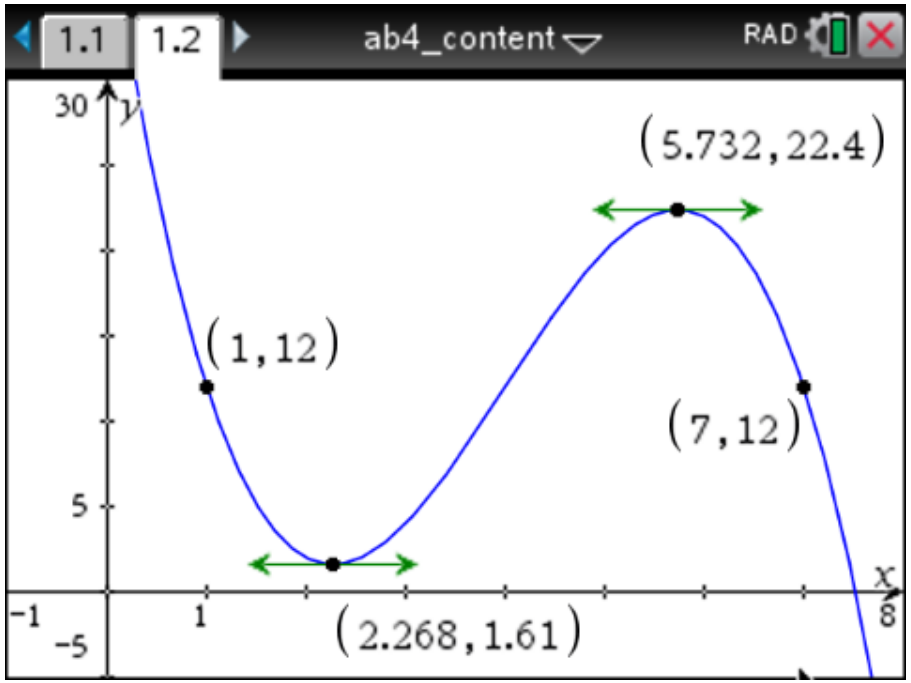
$$x = \frac{-24 \pm \sqrt{108}}{-6} = 4 \pm \sqrt{3}$$

$4 - \sqrt{3} = 2.268 = c_1 \qquad 4 + \sqrt{3} = 5.732 = c_2$

Solution

Here are the two values of c that satisfy the conclusion of Rolle's Theorem, and a graph that illustrates this solution.





Rolle's Theorem and Particle Motion

- Suppose a particle moves along a horizontal line so that its position at time t is given by the continuous and differentiable function $s(t)$.
- Suppose $s(a) = s(b)$: the particle is in the same place at two different times.
- Rolle's Theorem: There is some time $t = c$ between a and b such that $s'(c) = v(c) = 0$.
- There is some time $t = c$ where the velocity is 0.

Note: Use Rolle's Theorem to show there is only one real root in an interval.

The Mean Value Theorem

Let f be a function that satisfies the following conditions.

(1) f is continuous on the closed interval $[a, b]$.

(2) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

A Closer Look

(1) The MVT is an existence theorem.

(2) Rolle's Theorem is a special case of the Mean Value Theorem.

If $f(a) = f(b)$ then $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{0}{b - a} = 0$

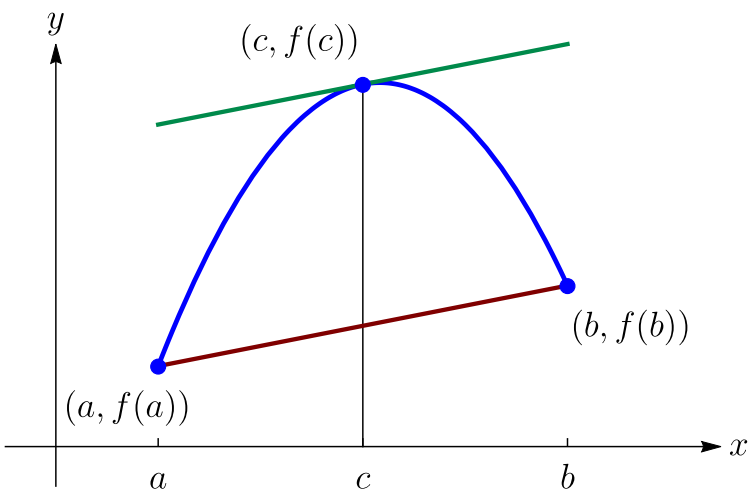
(3) MVT geometrically:

The slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$:

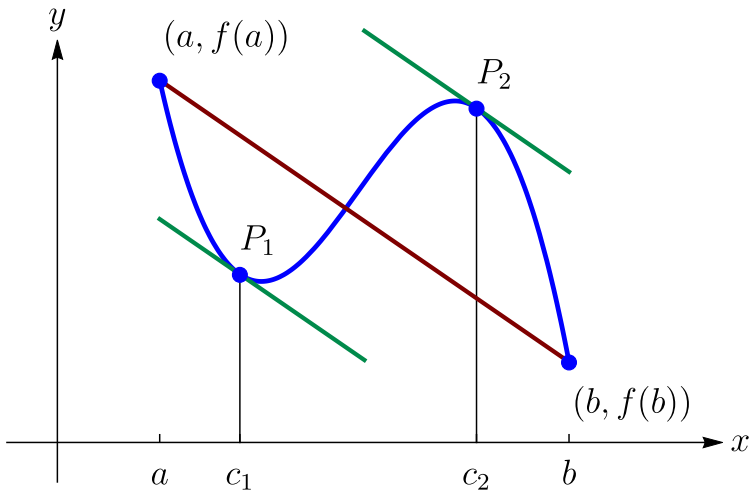
$$m = \frac{f(b) - f(a)}{b - a}$$

MVT: There is at least one point $(c, f(c))$ on the graph of f where the slope of the tangent line is the same as the slope of the secant line.

MVT Illustrations



The slope of the tangent line to the graph of f at $(c, f(c))$ is the same as the slope of the secant line connecting $(a, f(a))$ and $(b, f(b))$.



The tangent lines to the graph of f at P_1 and P_2 are parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Example 2 Fundamental MVT

Let $f(x) = \frac{1}{4}x^3 + \frac{1}{2}x^2 - 2x - 1$, $-4 \leq x \leq 3$.

Find all numbers c that satisfy the conclusion of the MVT.

Solution

f is a polynomial: continuous on $[-4, 3]$, differentiable on $(-4, 3)$.

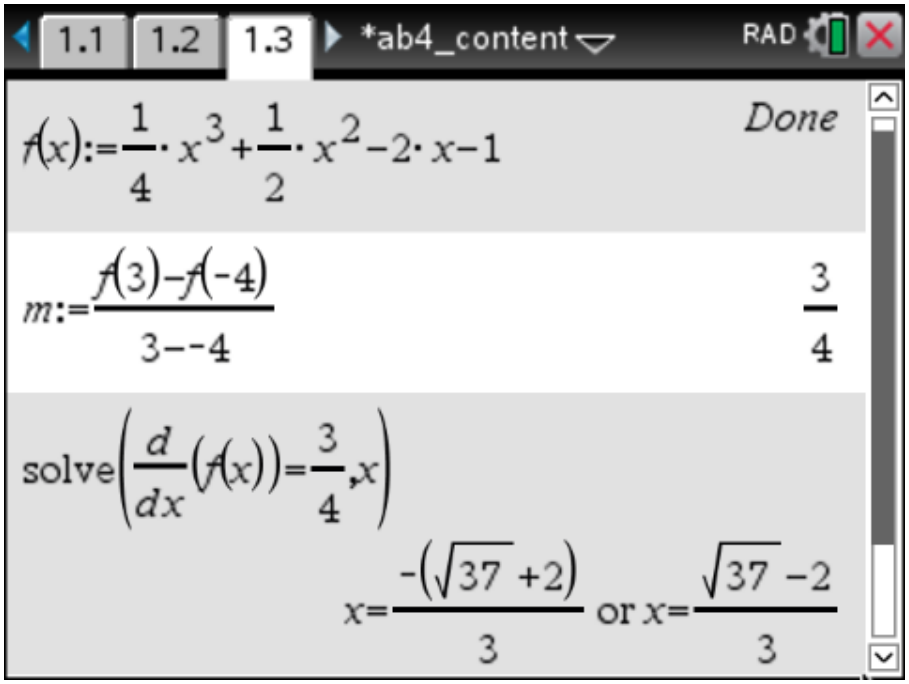
f satisfies the conditions of the MVT.

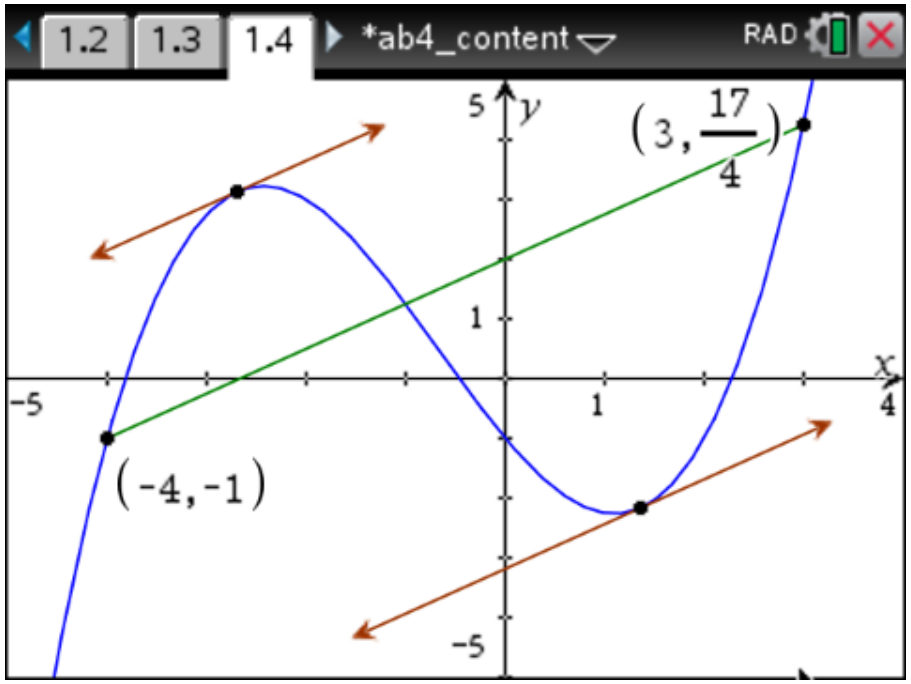
$$f'(x) = \frac{3}{4}x^2 + x - 2, \quad a = -4, \quad b = 3$$

$$f'(c) = \frac{3}{4}c^2 + c - 2 = \frac{f(3) - f(-4)}{3 - (-4)} = \frac{4.25 - (-1)}{7} = \frac{3}{4}$$

$$\frac{3}{4}c^2 + c - \frac{11}{4} = 0 \implies c = \frac{-2 \pm \sqrt{37}}{3}$$

Technology Solution





Other MVT Applications

(1) MVT and Particle Motion

Suppose a particle moves along a line so that its position at time t is given by $s(t)$.

The average velocity of the particle between $t = a$ and $t = b$: $\dfrac{s(b) - s(a)}{b - a}$

The velocity of the particle at $t = c$ is $s'(c)$.

MVT: At some time $t = c$ between a and b the instantaneous velocity $s'(c)$ is equal to the average velocity.

(2) MVT and a Maximum Value

Suppose $f(0) = 5$ and $f'(x) \leq 3$ for all values of x.
What is the largest value for $f(2)$?

Apply the MVT to the interval $[0, 2]$.

There exists a c such that $f(2) - f(0) = f'(c)(2 - 0)$

$f(2) = f(0) + 2f'(c) \leq 5 + 2 \cdot 3 = 11$

Theorem

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b), that is $f(x) = g(x) + C$ where C is a constant.

If two functions, f and g, have the same derivative,

(1) f and g must differ by a constant.

(2) Their graphs are vertical translations of each other. The graphs have the same shape, but are shifted up or down.

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