

TI in Focus: AP[®] Calculus

2019 AP[®] Calculus Exam: BC-5

Technology Solutions and Problem Extensions

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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions in greater detail
- (4) Solutions using technology
- (5) Problem Extensions

5. Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x) dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x) dx$ or show that it diverges.

$$(a) f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

3 : $\left\{ \begin{array}{l} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{array} \right.$

$$(b) \frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

3 : $\left\{ \begin{array}{l} 1 : \text{partial fraction} \\ \text{decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{array} \right.$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left(\frac{1}{6} \frac{1}{x-4} - \frac{1}{6} \frac{1}{x+2} \right) dx \\ &= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int_0^2 \frac{1}{x^2 - 2x + 1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\
 &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx \\
 &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right) \\
 &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right)
 \end{aligned}$$

Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

3 : $\begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

Part (a)

Solution

$$f(x) = \frac{1}{x^2 - 2x + k} = (x^2 - 2x + k)^{-1}$$

$$f'(x) = (-1)(x^2 - 2x + k)^{-2}(2x - 2) = \frac{(-1)(2x - 2)}{(x^2 - 2x + k)^2}$$

$$f'(0) = \frac{(-1)(2 \cdot 0 - 2)}{(0^2 - 2 \cdot 0 + k)^2} = \frac{2}{k^2} = 6$$

$$k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

Technology Solution

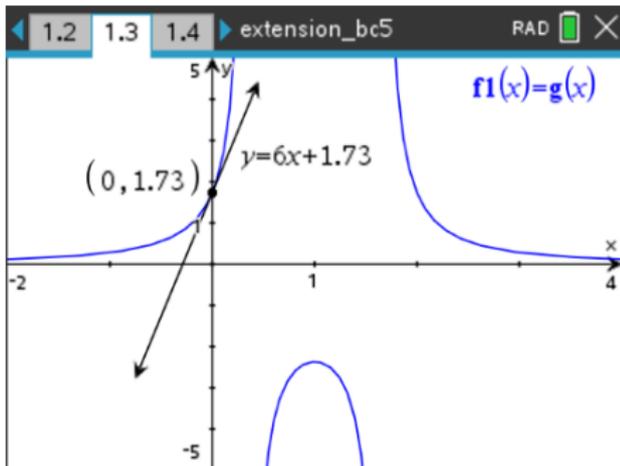
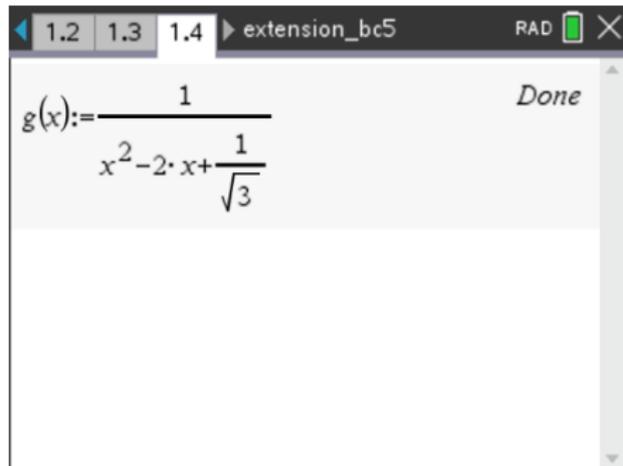
1.1 1.2 1.3 extension_bc5 RAD

$$f(x) := \frac{1}{x^2 - 2 \cdot x + k} \quad \text{Done}$$
$$\frac{d}{dx}(f(x)) = \frac{-2 \cdot (x-1)}{(x^2 - 2 \cdot x + k)^2}$$
$$\frac{d}{dx}(f(x))|_{x=0} = \frac{2}{k^2}$$

1.1 1.2 1.3 extension_bc5 RAD

$$\text{solve}\left(\frac{2}{k^2} = 6, k\right) | k > 0 \quad k = \frac{\sqrt{3}}{3}$$
$$\text{solve}\left(\left(\frac{d}{dx}(f(x))\right)|_{x=0} = 6, k\right) | k > 0 \quad k = \frac{\sqrt{3}}{3}$$

Graphical Presentation



Example 1 Runs in the Family

Consider the function g defined by $g(x) = \frac{1}{x^2 - 2x - 6}$.

- Find an equation for the tangent line to the graph of g at the value $x = 4$.
- Sketch a graph of g and the tangent line on the same coordinate axes.
- Find the value b where the tangent line intersects the x -axis.
- Find the area of the region bounded the graph of g , the tangent line, and the line $x = b$.

Part (b)

Solution

$$\int_0^1 f(x) dx = \int_0^1 \frac{1}{x^2 - 2x - 8} dx = \int_0^1 \frac{1}{(x-4)(x+2)} dx$$

Partial Fraction Decomposition

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} = \frac{A(x+2) + B(x-4)}{(x-4)(x+2)}$$

$$1 = A(x+2) + B(x-4)$$

$$x = 4: \quad 1 = A(6) \quad \Rightarrow \quad A = \frac{1}{6}$$

$$x = -2: \quad 1 = B(-6) \quad \Rightarrow \quad B = -\frac{1}{6}$$

Part (b)

Solution (Continued)

$$\int_0^1 f(x) dx = \int_0^1 \left[\frac{1/6}{x-4} - \frac{1/6}{x+2} \right] dx$$

Use the PFD

$$= \left[\frac{1}{6} \ln |x-4| - \frac{1}{6} \ln |x+2| \right]_0^1$$

Antiderivatives

$$= \left[\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right] - \left[\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right]$$

FTC

$$= -\frac{1}{6} [\ln 4 - \ln 2] = -\frac{1}{6} \ln \frac{4}{2} = -\frac{1}{6} \ln 2$$

Technology Solution

The screenshot shows a TI-84 Plus calculator interface with the following content:

- Top bar: 2.1 | 2.2 | 3.1 | *extensio...bc5 | RAD | [Green icon] | [Close icon]
- Function definition: $f(x) := \frac{1}{x^2 - 2 \cdot x - 8}$ Done
- Expansion: $\text{expand}(f(x))$ $\frac{1}{6 \cdot (x-4)} - \frac{1}{6 \cdot (x+2)}$
- Integration: $\int_0^1 f(x) dx$ $\frac{-\ln(2)}{6}$

Example 2 Unbounded Region

Consider the function g defined by $g(x) = \frac{1}{x^2 - 2x + 3}$.

- (a) Find the value of $\int_{-\infty}^{\infty} g(x) dx$ or show that it diverges.
- (b) Let R be the region in the first quadrant bounded by the graph of $y = g(x)$, the x -axis, and the y -axis. Find the volume of the solid obtained by rotating R about the x -axis.

Part (c)**Solution**

$$\begin{aligned}\int_0^2 \frac{1}{x^2 - 2x + 1} dx &= \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_0^b \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_b^2 \right) \\ &= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right)\end{aligned}$$

Part (c)

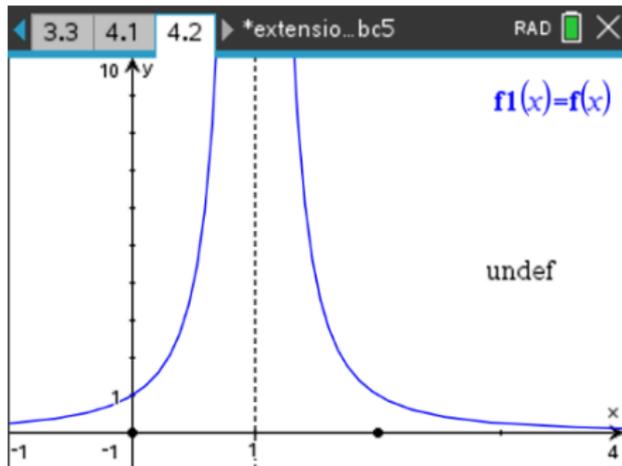
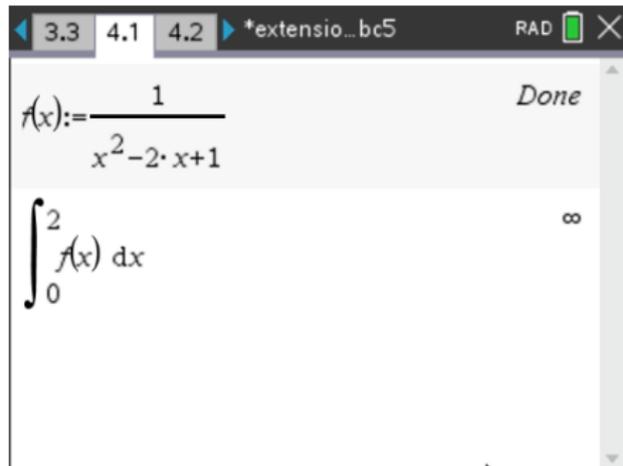
Solution (Continued)

$$\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right) = -\frac{1}{(-)} = +\infty$$

$$\lim_{b \rightarrow 1^+} \left(\frac{1}{b-1} \right) = \frac{1}{(+)} = \infty$$

Therefore, the (original) integral diverges.

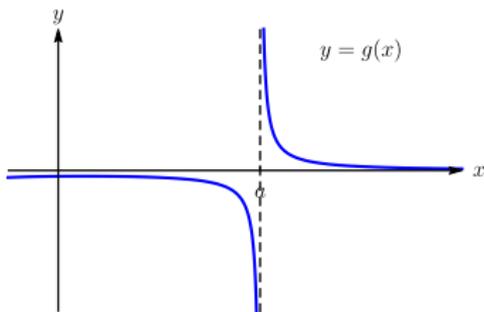
Technology Solution



Example 3

Consider the function g defined by $g(x) = \frac{1}{x^2 - x - 12}$.

A portion of the graph of g is shown in the figure.



- (a) Find the value of a such that $\lim_{x \rightarrow a^+} g(x) = \infty$
- (b) Let R be the region bounded by the graph of g , the x -axis, and the line $x = 3$. Find the area of the region R .
- (c) Find the value of $\int_{-4}^{-2} g(x) dx$ or show that it diverges.

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