TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-6
Scoring Guidelines

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Outline

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6. Functions f, g, and h are twice-differentiable functions with $g(2) = h(2) = 4$. The line $y = 4 + \frac{2}{3}(x - 2)$ is tangent to both the graph of g at $x = 2$ and the graph of h at $x = 2$.

(a) Find $h'(2)$.

(b) Let a be the function given by $a(x) = 3x^3 h(x)$. Write an expression for $a'(x)$. Find $a'(2)$.

(c) The function h satisfies $h(x) = \dfrac{x^2 - 4}{1 - (f(x))^3}$ for $x \neq 2$. It is known that $\lim_{x \to 2} h(x)$ can be evaluated using L'Hospital's Rule. Use $\lim_{x \to 2} h(x)$ to find $f(2)$ and $f'(2)$. Show the work that leads to your answers.

(d) It is known that $g(x) \leq h(x)$ for $1 < x < 3$. Let k be a function satisfying $g(x) \leq k(x) \leq h(x)$ for $1 < x < 3$. Is k continuous at $x = 2$? Justify your answer.

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(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

$3 : \begin{cases} 1 : \text{form of product rule} \\ 1 : a'(x) \\ 1 : a'(2) \end{cases}$

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(c) Because h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$.

Also, $\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3}$, so $\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4$.

Because $\lim_{x\to 2}\left(x^2-4\right) = 0$, we must also have $\lim_{x\to 2}\left(1-(f(x))^3\right) = 0$.

Thus, $\lim_{x\to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim_{x\to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus, $f'(2) = -\frac{1}{3}$.

$$4 : \begin{cases} 1 : \lim_{x\to 2} \frac{x^2-4}{1-(f(x))^3} = 4 \\ 1 : f(2) \\ 1 : \text{L'Hospital's Rule} \\ 1 : f'(2) \end{cases}$$

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(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x \to 2} g(x) = g(2) = 4$ and $\lim_{x \to 2} h(x) = h(2) = 4$. Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x \to 2} k(x) = 4$. Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$. Thus, k is continuous at $x = 2$.	1 : continuous with justification

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Student Performance

Part (a)

- Most students recognized that $h'(2)$ is the slope of the tangent line to the graph of $y = h(x)$ at $x = 2$ and, therefore, $h'(2) = 2/3$.
- Some students identified the y-intercept of the tangent line as the slope: $h'(2) = 4$.

Part (b)

- Most students recognized the need to use the Product Rule.
- Some common mistakes: copy errors, parentheses errors, incorrect derivative of $3x^3$.
- Most students knew to evaluate $a'(2)$; some simplification errors.

Student Performance

Part (c)

- Few students indicated that h is continuous at $x = 2$.
- Many students did not connect $\lim_{x \to 2} h(x)$ to $h(2) = 4$.
- Most students understood the need to apply L'Hospital's Rule; some errors in derivatives.
- Many were able to find $f(2)$, but fewer students were able to find $f'(2)$.
- Most common error: $\lim_{x \to 0} h(x) = \frac{0}{0}$.

Part (d)

- Few students presented a complete justification to earn the point in this part.
- Most common errors: failure to mention that g and h are continuous,
  no discussion of $\lim_{x \to 2} k(x)$,
  incorrect theorem: MVT or IVT.

Part (a) 1: answer

(1) Must be our answer: $\frac{2}{3}$.

(2) Unsupported or unlabeled $\frac{2}{3}$: OK

(3) If included as part of a tangent line equation, must be boxed or circled.

(4) $y' = \frac{2}{3}$: OK

(5) Most common incorrect answer: $h'(2) = 4$.

Part (b) 1: form of the product rule

(1) Expression does not need to be labeled as $a'(x)$.

(2) Answer must be of the form:
$a'(x) = p(x) \cdot h(x) + q(x) \cdot h'(x)$
Where $p(x)$ and $q(x)$ are polynomials.

(3) Form of product rule:

- Sum of two terms.
- $h(x)$ in one term.
- $h'(x)$ in the other term.
- Both $h(x)$ and $h'(x)$ multiplied by polynomials.

Part (b) 1: $a'(x)$

(1) Expression presented does not need to be labeled as $a'(x)$.

(2) Common alternate correct form: $a'(x) = 3[x^3h'(x) + 3x^2h(x))]$

Examples

(1) $a'(x) = 3x^2h'(x)$ 0 - 0 - 0

(2) $a'(x) = 9x^2h(x) + 3x^2h'(x)$ 1 - 0 - ?

(3) $a'(x) = 3x^3h'(x) + h(x) + 9x^2$ 0 - 0 - 0

(4) $a'(x) = 9x^2[h(x) + 3x^2h'(x)]$ 1 - 0 - ?

Part (b) 1: $a'(2)$

(1) Eligibility: earned first point;
Unless $y = 4 + \frac{2}{3}(x-2)$ is used for $h(x)$.

(2) If $a'(x)$ is incorrect, read for consistent answer.

(3) If $h'(2)$ is incorrect (from part (a)), read for a consistent answer.

(4) $a'(x) = a'(2)$ does not earn this point (linkage error).

(5) Many unsimplified forms OK.

Examples

(1) $a'(x) = 6x^2h(x) + 3x^3h'(x) \quad \Rightarrow \quad a'(2) = 112$ 1 - 0 - 1

(2) $a'(x) = 3x^2h'(x) + 6h(x) \quad \Rightarrow \quad a'(2) = 56$ 1 - 0 - 1

Other Common Errors

Imported $h'(2)$ from part (a); $h'(2) = 4$; arithmetic errors in evaluating $a'(2)$.

Part (c) 1: $\lim_{x \to 2} \frac{x^2-4}{1-(f(x))^3} = 4$

(1) This expression often appears later in the problem.

(2) Can appear after applying L'Hospital's Rule.

(3) Connects $\lim_{x \to 2} h(x)$ to 4 or
Connects a correct evaluation of $\lim_{x \to 2} h(x)$ to 4.

(4) Does not need to mention $h(x)$ is continuous.

Part (c) Examples

(1) $\lim_{x \to 2} h(x) = 4$ — 1 - ? - ? - ?

(2) $\lim_{x \to 2} \dfrac{2x}{-3 \cdot f(x)^2 \cdot f'(x)} = 4$ — 1 - ? - 1 - ?

(3) $\dfrac{2 \cdot 2}{-3(f(2)^2 \cdot f'(2)} = 4$ — 1 - ? - ? - ?

(4) $\lim_{x \to 2} \dfrac{2x}{3 \cdot f(x)^2 \cdot f'(x)} = 4$ — 0 - ? - 1 - ?

(5) $\dfrac{2^2 - 4}{1 - f(2)^3} = 4$ — 0 - ? - ? - ?

(6) $\dfrac{4}{-3 \cdot f'(2)} = 4$ — 0 - ? - ? - ?

Part (c) 1: $f(2)$

(1) No bald answers.

(2) Philosophy: sets our denominator equal to 0.

(3) Does not need to mention L'Hospital's Rule.

(4) Must be our answer: $f(2) = 1$.

Examples

(1) $1 - f(2)^3 = 0 \quad \Rightarrow \quad f(2) = 1$? - 1 - ? - ?

(2) $1 = f(2)^3 \quad \Rightarrow \quad f(2) = 1$? - 1 - ? - ?

(3) $4\left[1 - f(2)^3\right] = 0 \quad \Rightarrow \quad f(2) = 1$? - 0 - ? - ?

(4) $4 - 4 \cdot f(2)^3 = 0 \quad \Rightarrow \quad f(2) = 1$? - 0 - ? - ?

Part (c) 1: L'Hospital's Rule

Notation

(1) $\lim_{x\to 2} \dfrac{x^2-4}{1-f(x)^3} = \dfrac{0}{0}$

Does not earn third point; eligible for fourth point.

(2) These expression may be presented, and student may go on to earn the third point.

- $\lim_{x\to 2} \dfrac{x^2-4}{1-f(x)^3} \quad \Rightarrow \quad \dfrac{0}{0}$
- $\lim_{x\to 2} \dfrac{x^2-4}{1-f(x)^3} \quad \begin{matrix} \to \\ \to \end{matrix} \quad \dfrac{0}{0}$
- $\lim_{x\to 2} \dfrac{x^2-4}{1-f(x)^3} \quad \dfrac{0}{0}$

Part (c) 1: L'Hospital's Rule

(1) Philosophy: limit of a quotient of derivatives.

(2) Limit

(3) Quotient.

(4) Convinced differentiation was attempted by seeing $f'(x)$ in denominator.

Part (c) 1: $f'(2)$

(1) Eligibility: must have earned the third point, unless

- Quotient was correct but limit was missing, or
- $\lim_{x \to 2} h(x) = \frac{0}{0}$

(2) Must be our answer.

(3) Cannot import an incorrect $f(2)$.

Part (c) Examples

(1) $\lim_{x \to 2} \dfrac{2x}{-3 \cdot f(x)^2 \cdot f'(x)}$? - ? - 1 - ?

(2) $\lim_{x \to 2} \dfrac{2x}{3 \cdot f(x)^2 \cdot f'(x)}$? - ? - 1 - 0

(3) $\lim_{x \to 2} \dfrac{2x - 4}{1 - 3 \cdot f(x)^2 \cdot f'(x)}$? - ? - 1 - 0

(4) $\dfrac{2x}{-3 \cdot f(x)^2 \cdot f'(x)}$? - ? - 0 - ?

(5) $\dfrac{2x}{3 \cdot f(x)^2 \cdot f'(x)}$? - ? - 0 - 0

(6) $\lim_{x \to 2} \dfrac{x^2 - 4}{1 - f(x)^3} = \dfrac{0}{0}$? - ? - 0 - ?

Part (d) 1: continuous with justification

(1) Alternate names for the Squeeze Theorem

- Sandwich Theorem
- Pinching Theorem
- Squeeze Play Theorem
- Flyswatter Theorem

(2) Not Alternate names for the Squeeze Theorem

- Intermediate Value Theorem
- Mean Value Theorem
- Extreme Value Theorem
- Rolle's Theorem

Part (d) 1: continuous with justification

Need to see:

(1) Yes

(2) $g(x)$ and $h(x)$ are continuous

(3) $\lim_{x \to 2} g(x) = 4 = \lim_{x \to 2} h(x)$

(4) Therefore, $\lim_{x \to 2} k(x) = 4$

(5) $k(2) = 4$ because $g(2) = 4$ and $h(2) = 4$

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