TI in Focus: AP® Calculus

2020 Mock AP® Calculus Exam

AB-1: Solutions, Concepts, and Scoring Guidelines

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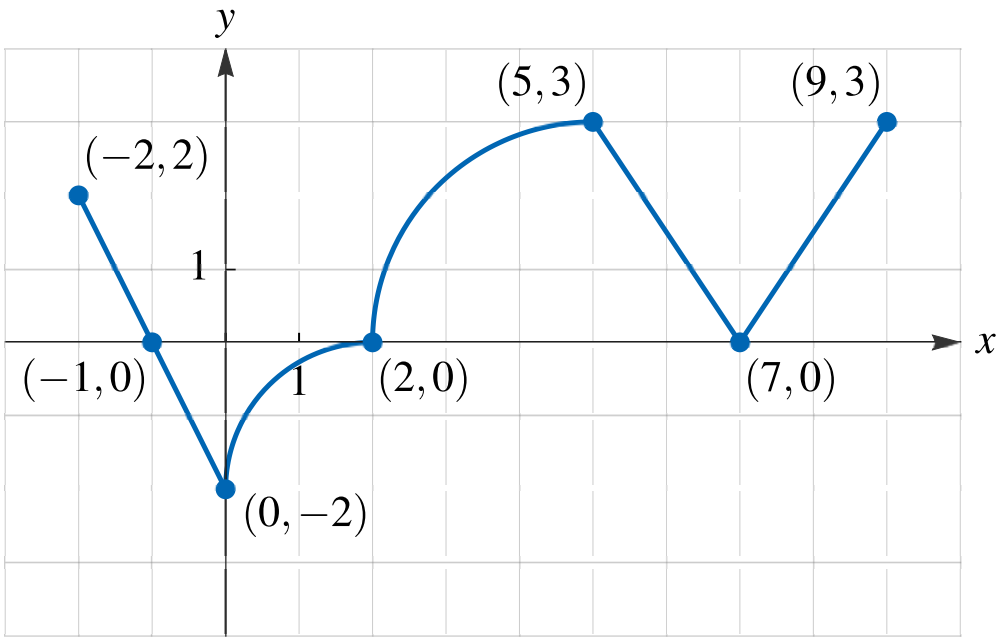
AB 1

The continuous function f has domain $-2 \leq x \leq 9$. The graph of f, consisting of three line segments and two quarter circles, is shown in the figure.



Graph of f

Let g be the function defined by $g(x) = \int_0^x f(t)\,dt$ for $-2 \leq x \leq 9$.

(i) Find $\lim_{x \to 0} \frac{3x + g(x)}{\sin x}$. Show the computations that lead to your answer.

Key Concepts

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

That is, if the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is in an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

A Closer Look

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are met.

   It is very important to verify the conditions regarding the limits of f and g before using L'Hospital's Rule. That means, you can only apply this rule if the limit is in an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

2. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. That means $x \to a$ can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

3. This is an amazing technique, but many limits may be determined using other, often simpler, methods.

4. L'Hospital's Rule may be used repeatedly to evaluate a limit in an appropriate indeterminate form.

5. There are other indeterminate forms:

   $0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$

Solution

$$\lim_{x\to 0}[3x+g(x)] = \lim_{x\to 0} 3x + \lim_{x\to 0} g(x)$$

$$= 3\cdot 0 + g(0) = 0 + \int_0^0 f(t)\,dt = 0 + 0 = 0$$

$$\lim_{x\to 0} \sin x = \sin 0 = 0$$

The limit $\displaystyle\lim_{x\to 0} \frac{3x+g(x)}{\sin x}$ is in an indeterminate form of type $\dfrac{0}{0}$.

We can use L'Hospital's Rule.

Solution (Continued)

$$\lim_{x\to 0} \frac{3x + g(x)}{\sin x} = \lim_{x\to 0} \frac{3 + g'(x)}{\cos x}$$ L'Hospital's Rule

$$= \lim_{x\to 0} \frac{3 + f(x)}{\cos x}$$ Fundamental Theorem of Calculus

$$= \frac{3 + f(0)}{\cos 0} = \frac{3 + (-2)}{1} = 1$$ Evaluate limit

Scoring Guidelines

(i) $\lim_{x\to 0}(3x + g(x)) = 0$

$\lim_{x\to 0} \sin x = 0$

Therefore the limit $\lim_{x\to 0} \dfrac{3x + g(x)}{\sin x}$ is in the indeterminate form $\dfrac{0}{0}$ and L'Hospital's Rule can be applied.

$$\lim_{x\to 0} \frac{3x + g(x)}{\sin x} = \lim_{x\to 0} \frac{3 + g'(x)}{\cos x} = \frac{3 + g'(0)}{\cos 0}$$

$$= \frac{3 + f(0)}{\cos 0} = \frac{3 + -2}{1} = 1$$

$$3 : \begin{cases} 1 : \text{conditions for L'Hospital's Rule} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- The first point is earned for conveying that both the numerator and the denominator each approach 0.
- The response does not need to reference continuity or differentiability.
- Any notation in which limit $= \frac{0}{0}$ does not earn the first point.

  This can appear in *disguise*.
- Other presentations are OK: $\to \frac{0}{0}$, $=$ indeterminate form $\frac{0}{0}$.
- The second point is earned by presenting a limit of a fraction whose numerator and denominator are clearly attempts at derivatives.
- The second point may be earned even if the limit notation is missing from some expressions.
  But the second point cannot be earned unless the response conveys a limit.
- The third point is earned for the correct answer 1.

(j) The function h is defined by $h(x) = x \cdot g(x^2)$. Find $h'(\sqrt{2})$. Show the computations that lead to your answer.

Key Concepts

The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)]$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x)$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

A Closer Look

1. In words: the derivative of the composition of two functions is the product of the derivative of the *outer* function evaluated at the *inner* function and of the derivative of the inner function.

2. Other notation: $(f \circ g)' = f'(g(x)) \cdot g'(x)$

3. And more notation: $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

4. Given $(f \circ g)(x)$, here is a procedure for finding $(f \circ g)'(x)$ using the Chain Rule.
   (a) Identify $f(x)$ and $g(x)$.
   (b) Find $f'(x)$ and $g'(x)$.
   (c) Write the final answer as $f'(g(x))\, g'(x)$.

Solution

$$h'(x) = 1 \cdot g(x^2) + x \cdot g'(x^2) \cdot 2x$$ Product Rule; Chain Rule.

$$= g(x^2) + 2x^2 f(x^2)$$ Simplify; $g' = f$

$$h'(\sqrt{2}) = g(2) + 2 \cdot 2 \cdot f(2)$$ Evaluate $h'(\sqrt{2})$.

$$= (\pi - 4) + 4 \cdot 0 = \pi - 4$$ Use values for $g(2)$ and $f(2)$; simplify.

Scoring Guidelines

(j) $h'(x) = 1 \cdot g(x^2) + x \cdot g'(x^2) \cdot 2x$

$= g(x^2) + 2x^2 f(x^2)$

$h'(\sqrt{2}) = g(2) + 2 \cdot 2 \cdot f(2)$

$= (\pi - 4) + 4 \cdot 0 = \pi - 4$

$$3 : \begin{cases} 1 : \text{product rule} \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- The first point is earned for an expression of the form: $a \cdot g(x^2) + b \cdot x \cdot g'(x^2)$
- The second point is earned for $2x \cdot g'(x^2)$ as part of one term in $h'(x)$.
- The third point is earned for the correct answer, with supporting work.
- Minimal computations: $g(2) + 4g'(2) = \pi - 4$
- Must evaluate g and g' to earn the third point.
- Eligibility for the third point: must earn the first two points.

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