

TI in Focus: AP[®] Calculus

2021 AP[®] Calculus Exam: BC-2

Particle Motion

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Outline

- (1) Velocity, speed
- (2) Distance traveled, acceleration
- (3) Position
- (4) Examples

Particle Motion in the Plane

Suppose a particle moves in the plane so that its position at time t is given by the parametric equations $x = f(t)$ and $y = g(t)$.

Consider the vector function $\mathbf{r} = \langle f(t), g(t) \rangle$.

$\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t))$.

If $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ then $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$

The **velocity vector** $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

The **speed** of the particle at time t is the magnitude of the velocity vector.

$$\text{speed} = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Particle Motion in the Plane (Continued)

The total distance traveled by the particle from time $t = \alpha$ to time $t = \beta$ is

$$\text{distance traveled} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

The **acceleration** of the particle at time t is defined to be the derivative of the velocity.

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

Use the Fundamental Theorem of Calculus to find the x - and y -coordinates at time $t = t_1$.

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} x'(t) dt \quad y(t_1) = y(t_0) + \int_{t_0}^{t_1} y'(t) dt$$

Example 1 Particle Motion

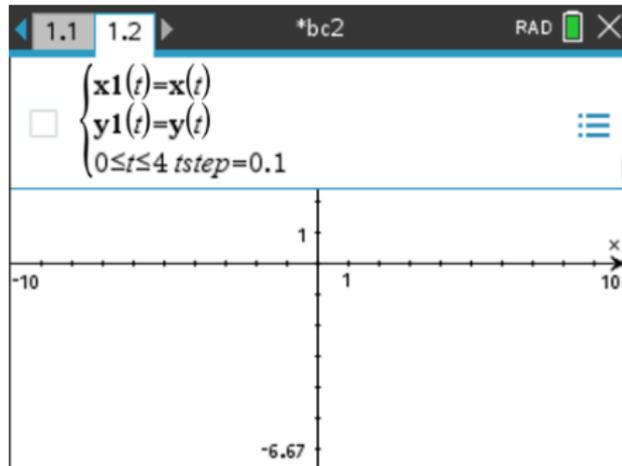
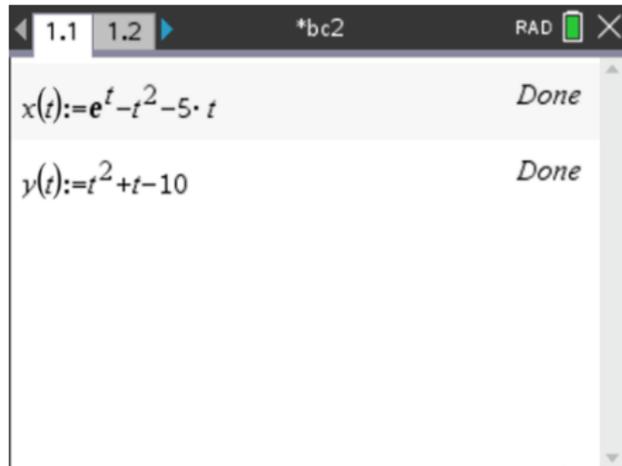
A particle moving along a curve in the xy -plane is a position $(x(t), y(t))$ at time t , for $t \geq 0$, where

$$x(t) = e^t - t^2 - 5t \quad y(t) = t^2 + t - 10$$

- (a) Sketch the path of the particle.
- (b) Find the velocity vector and the acceleration vector at time $t = 1$.
- (c) Find the speed of the particle at time $t = 3$.
- (d) When is the particle farthest to the left? What is the position of the particle at this time?

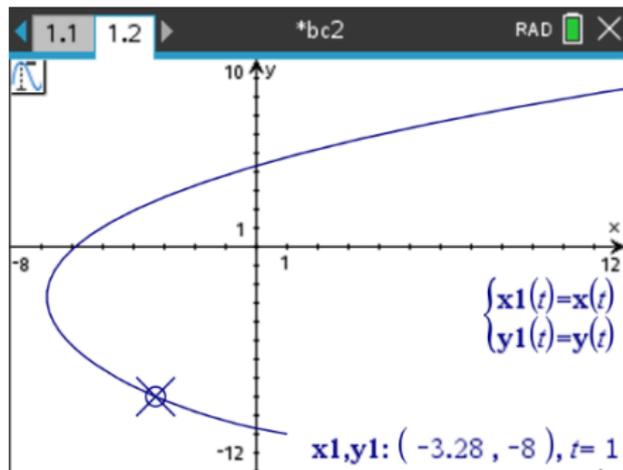
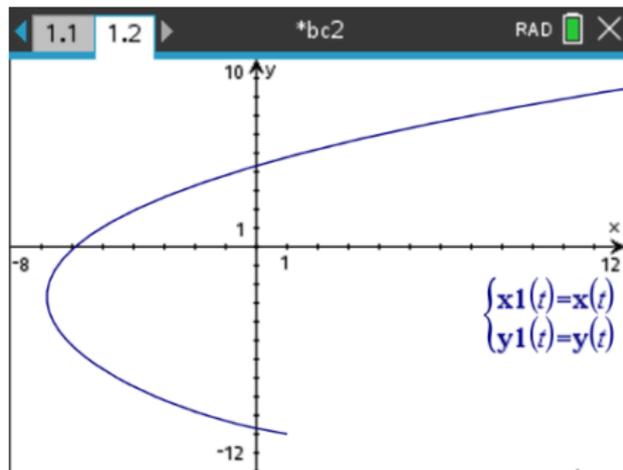
Solution

(a) Sketch the path of the particle.



Solution

(a) Sketch the path of the particle.



Solution

(b) Velocity vector and acceleration vector at time $t = 1$.

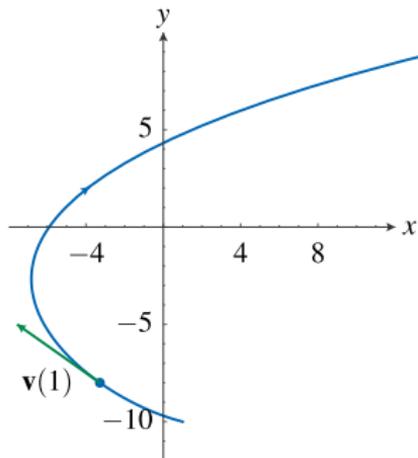
Find the component functions of the velocity vector, and evaluate at $t = 1$.

$$x'(t) = e^t - 2t - 5 \Rightarrow x'(1) = e^1 - 2(1) - 5 = e - 7$$

$$y'(t) = 2t + 1 \Rightarrow y'(1) = 2(1) + 1 = 3$$

The velocity vector is $\mathbf{v}(1) = \langle e - 7, 3 \rangle$

Note: the velocity vector is also the tangent vector, and points in the direction of the tangent line.



The position of the particle at time $t = 1$:

$$\begin{aligned}(x(1), y(1)) \\ = (e - 6, -8) = (-3.282, -8)\end{aligned}$$

Solution

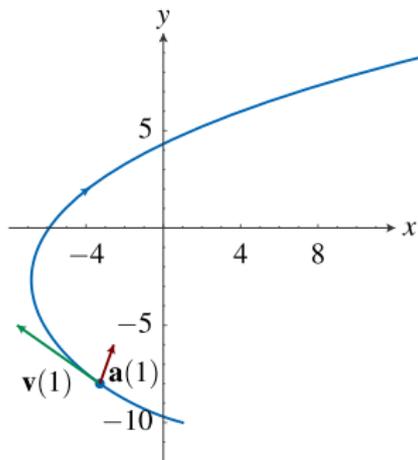
(b) Velocity vector and acceleration vector at time $t = 1$.

Find the component functions of the acceleration vector, and evaluate at $t = 1$.

$$x''(t) = e^t - 2 \Rightarrow x''(1) = e - 2$$

$$y''(t) = 2 \Rightarrow y''(1) = 2$$

The acceleration vector is $\mathbf{a}(1) = \langle e - 2, 2 \rangle$



Solution

(b) Velocity vector and acceleration vector at time $t = 1$.

TI-84 Plus calculator screenshot showing the calculation of velocity components at $t = 1$. The window title is *bc2 and the mode is RAD. The cursor is on the 1.3 tab.

$x_p(t) := \frac{d}{dt}(x(t))$ Done
 $y_p(t) := \frac{d}{dt}(y(t))$ Done
 $[x_p(1) \ y_p(1)]$ $[e-7 \ 3]$

TI-84 Plus calculator screenshot showing the calculation of acceleration components at $t = 1$. The window title is *bc2 and the mode is RAD. The cursor is on the 1.4 tab.

$x_{pp}(t) := \frac{d^2}{dt^2}(x(t))$ Done
 $y_{pp}(t) := \frac{d^2}{dt^2}(y(t))$ Done
 $[x_{pp}(1) \ y_{pp}(1)]$ $[e-2 \ 2]$

Solution

(c) Find the speed of the particle at time $t = 3$.

$$\text{speed} = \sqrt{[x'(3)]^2 + [y'(3)]^2}$$

Equation for speed

$$= \sqrt{[e^3 - 2(3) - 5]^2 + [2(3) + 1]^2}$$

Use expressions for $x'(t)$ and $y'(t)$

$$= \sqrt{(e^3 - 11)^2 + 49} = 11.469$$

Simplify

The image shows a TI-84 Plus calculator screen. The top status bar displays '1.3', '1.4', '1.5', '*bc2', 'RAD', and a green battery icon. The main display area shows the expression $\sqrt{(xp(3))^2 + (yp(3))^2}$ on the left and the numerical result $\sqrt{e^6 - 22 \cdot e^3 + 170}$ on the right. Below this, the same expression is shown again, followed by the decimal result 11.4694.

Solution

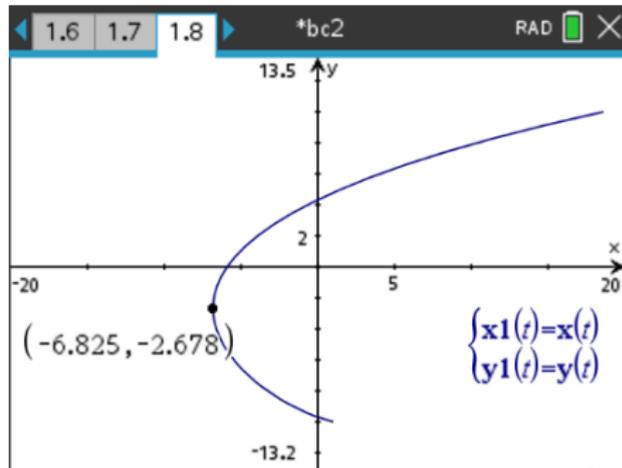
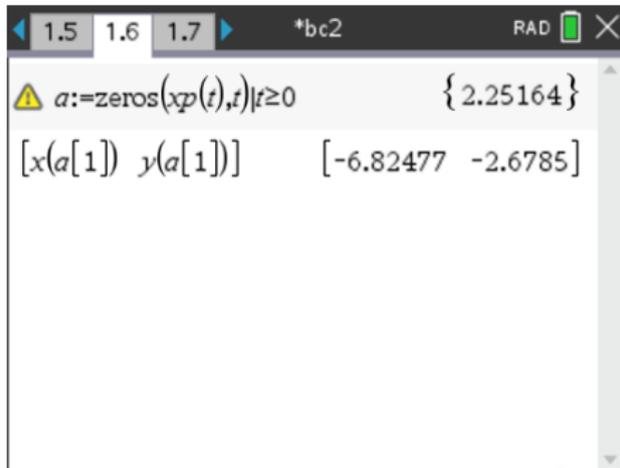
- (d) When is the particle farthest to the left? What is the position of the particle at this time?

$$x'(t) = e^t - 2t - 5 = 0 \Rightarrow t = 2.252$$

For $0 \leq t < 2.252$, $x'(t) < 0$ and for $t > 2.252$, $x'(t) > 0$.

Therefore, the particle is farthest to the left when $t = 2.252$.

It's position at that time is $(x(2.252), y(2.252)) = (-6.825, -2.679)$



Example 2 Travel Plans

A particle moving along a curve in the xy -plane is at position $(x(t), y(t))$ at time t , for $t \geq 0$, where

$$\frac{dx}{dt} = e^{\sin 3t} \cdot \tan^{-1}(e^{-t}) \quad \frac{dy}{dt} = t\sqrt{4t^3 - t^2 + 3}$$

- Find the speed of the particle at time $t = 1$.
- Find the distance traveled by the particle over the time interval $1 \leq t \leq 4$.
- Find the average speed of the particle over the time interval $1 \leq t \leq 4$.

Solution

(a) Find the speed of the particle at time $t = 1$.

$$\text{speed} = \sqrt{[e^{\sin 3} \tan^{-1}(e^{-1})]^2 + [1 \cdot \sqrt{4 \cdot 1 - 1^2 + 3}]^2} = 2.483$$

*bc2		RAD	✕
$x_p(t) := e^{\sin(3 \cdot t)} \cdot \tan^{-1}(e^{-t})$		Done	
$y_p(t) := t \cdot \sqrt{4 \cdot t^3 - t^2 + 3}$		Done	
$\sqrt{(x_p(1.))^2 + (y_p(1.))^2}$	2.4829		
$2.4828991893396 \rightarrow \text{speed}$	2.4829		

Solution

(b) Find the distance traveled by the particle over the time interval $1 \leq t \leq 4$.

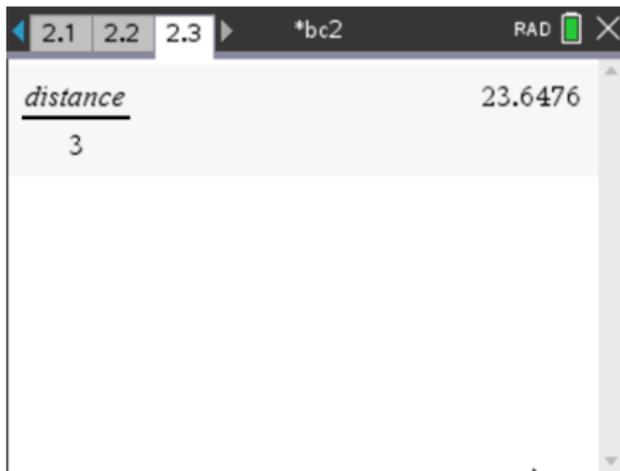
$$\text{distance traveled} = \int_1^4 \sqrt{[e^{\sin 3t} \cdot \tan^{-1}(e^{-t})]^2 + [t\sqrt{4t^3 - t^2 + 3}]^2} dt = 70.943$$

The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is '*bc2' and the mode is 'RAD'. The main display shows the integral expression $\int_1^4 \sqrt{(x_p(t))^2 + (y_p(t))^2} dt$ with a yellow warning triangle to the left. The result of the integration is 70.9429. Below the main display, the text '70.942926715245 → distance' is visible, followed by the value 70.9429.

Solution

(c) Find the average speed of the particle over the time interval $1 \leq t \leq 4$.

$$\text{ave speed} = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{70.943}{3} = 23.648$$



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