Concepts

If a function f has derivatives of all orders, then under certain conditions we can write f as a Taylor series centered at a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In the case where $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

This is called a Maclaurin series.

This expression means that $f(x)$ is the limit of the sequence of partial sums. The partial sums are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

The expression for $T_n(x)$ is a polynomial of degree n. $T_n(x)$ is called the nth-degree Taylor polynomial of f at $x = a$, or centered at a.

Let $R_n(x) = f(x) - T_n(x)$ so that $f(x) = T_n(x) + R_n(x)$, then $R_n(x)$ is called the remainder of the Taylor series. If it is possible to show that $\lim_{n\to\infty} R_n(x) = 0$ then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

Course and Exam Description Unit

Section 10.11: Finding Taylor Polynomial Approximations of Functions

Section 10.14: Finding Taylor or Maclaurin Series for a Function

Calculator Files

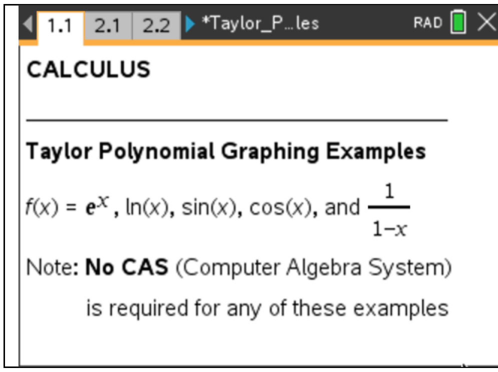
Taylor_Polynomial_Examples.tns

Using the Document

This tns file is used to produce the graphs of various Taylor polynomials, $y = T_n(x)$. These graphs are used to study how the accuracy of a Taylor polynomial is associated with the degree of the Taylor polynomial. The accuracy of each Taylor polynomial is visualized and can be related to symmetry, arc length, and any points of discontinuity.

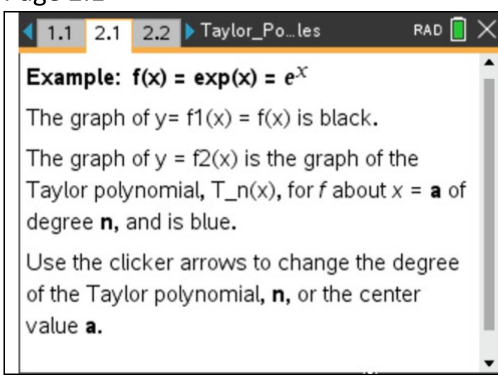
Five common functions are considered and some specific questions about associated Taylor polynomials are included.

Page 1.1



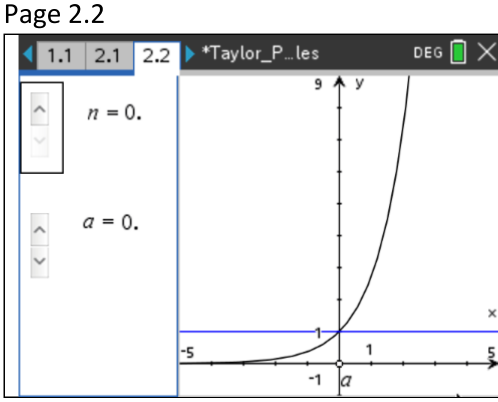
This opening page lists the five functions considered in this tns file: $f(x) = e^x$, $f(x) = \ln x$, $f(x) = \sin x$, $f(x) = \cos x$, and $f(x) = \dfrac{1}{1-x}$. Note that a CAS calculator is not required for any of these examples.

Page 2.1



This page describes the first example, the notation used, and the method for changing the values of n and a. In Problem 2 of this tns file, $f(x) = e^x$ and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.

Page 2.2



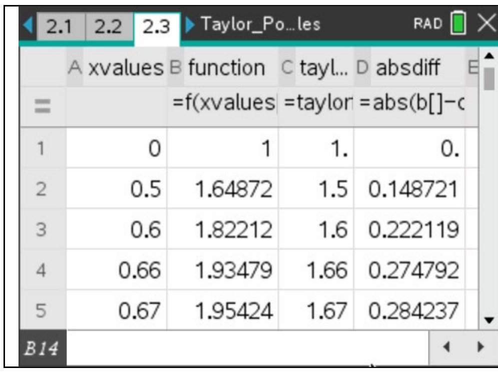
The graph of $y = f(x)$ is black and the graph of $y = T_n(x)$ is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are $n = 0$ and $a = 0$.

Note that it is also possible to grab the point a on the x-axis and move it horizontally to change the value.

Problems

1. For $a = 0$, sketch and describe the graph of $y = T_1(x)$.
2. Use the graph of $y = T_1(x)$ and the Trace All feature to describe the accuracy of the Taylor polynomial approximation as x moves farther away from $a = 0$.
3. Set $n = 2$. Sketch and describe the graph of $y = T_2(x)$.
4. Set $n = 3$. Sketch and describe the graph of $y = T_3(x)$.
5. Consider the graph of other Taylor polynomials for $n \geq 4$. Describe the accuracy of the Taylor polynomial approximation as n increases.

Page 2.3



	A xvalues	B function	C tayl...	D absdiff
=		=f(xvalues	=taylor	=abs(b[]−c
1	0	1	1.	0.
2	0.5	1.64872	1.5	0.148721
3	0.6	1.82212	1.6	0.222119
4	0.66	1.93479	1.66	0.274792
5	0.67	1.95424	1.67	0.284237

This is a Lists and Spreadsheet page in which values for x can be entered in column A. The following values are automatically computed: $f(x)$, $T_n(x)$, and $|f(x) - T_n(x)|$ (an indication of the accuracy of the approximation), in columns B, C, and D, respectively. These values are dependent on the current values of n and a.

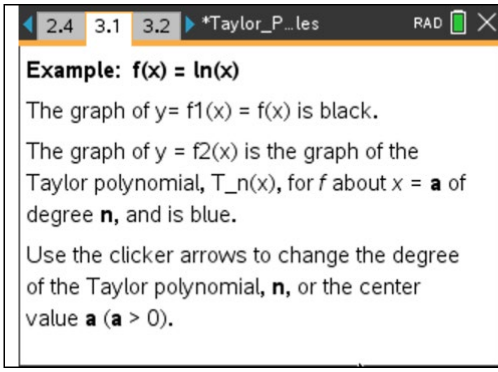
Problems

Change the values of n and a as necessary and use Page 2.3, the Lists and Spreadsheet page, to answer the following questions.

1. For a fixed value of n, describe the accuracy of the Taylor polynomial approximation as the values of x are farther away from a.
2. For fixed values of a and x, describe the accuracy of the Taylor polynomial approximation as n increases.
3. Set $a = 0$. For $n = 1, 2, 3, 4, 5$, find an interval in which the Taylor polynomial is a good approximation for f.

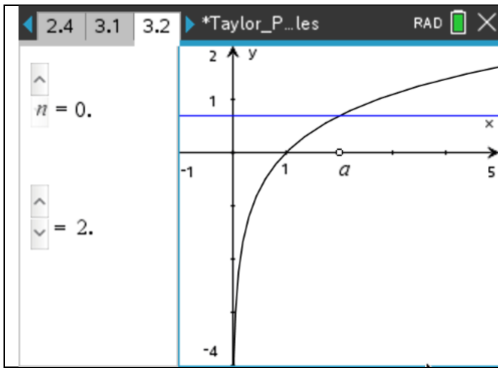
Note: Page 2.4 is used for background calculations. The equations and entries on this Lists and Spreadsheet page should be left unchanged to ensure the accuracy of the results presented on other pages of this problem.

Page 3.1



This page describes the second example, the notation used, and the method for changing the values of n and a. In Problem 3 of this tns file, $f(x) = \ln x$ and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value. Note that the value of a must be greater than 0 in this example.

Page 3.2



The graph of $y = f(x) = \ln x$ is black and the graph of $y = T_n(x)$ is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are $n = 0$ and $a = 2$.

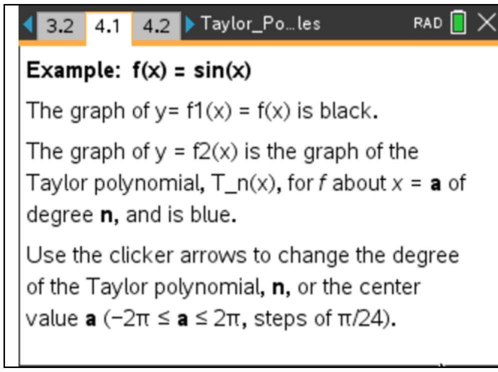
Note that it is also possible to grab the point a on the x-axis and move it horizontally to change the value.

Problems

Change the values of n and a as necessary and use Page 3.2 to answer the following questions.

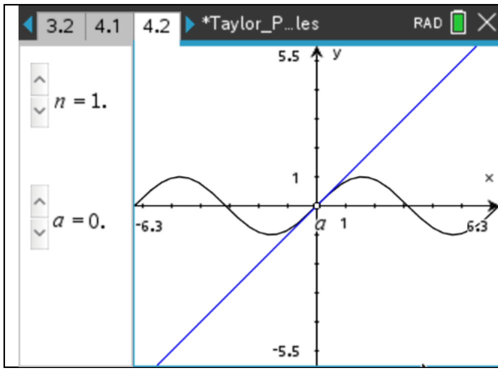
1. For $a = 2$, describe the accuracy of the Taylor polynomial approximation as n increases.
2. Describe the behavior of each Taylor polynomial as $x \to +\infty$ and as $x \to -\infty$. Explain how the graph of the Taylor polynomial changes, as $x \to +\infty$, as n increases by 1, for example, from $n = 6$ to $n = 7$. Explain why this property of the Taylor polynomials alternates as n increases.
3. Set $a = 0.3$. Consider the graph of $y = T_n(x)$ for various values of n. Explain why the Taylor polynomial appears to be a very good approximation to the left of $a = 0.3$ but diverges rapidly to the right of $a = 0.3$.

Page 4.1



This page describes the third example, the notation used, and the method for changing the values of n and a. In Problem 4 of this tns file, $f(x) = \sin x$ and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.
Note that in this example, the value of a must be in the interval $[-2\pi, 2\pi]$ and can be changed in increments of $\pi/24$.

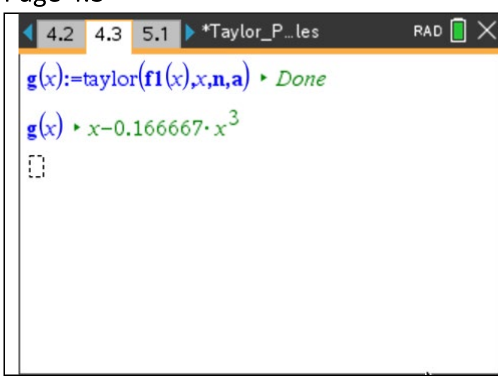
Page 4.2



The graph of $y = f(x) = \sin x$ is black and the graph of $y = T_n(x)$ is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are $n = 1$ and $a = 0$.

Note that it is also possible to grab the point a on the x-axis and move it horizontally to change the value.

Page 4.3



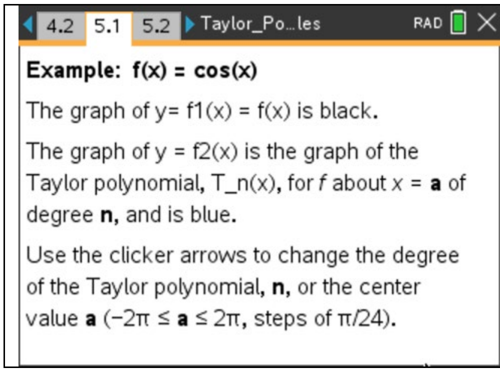
This Notes page defines and displays the Taylor polynomial $g(x) = T_n(x)$ for the current values of a and n.

Problems

Change the values of n and a as necessary and use Pages 4.2 and 4.3 to answer the following questions.

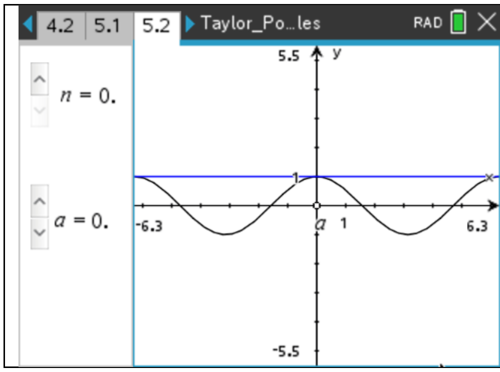
1. Set $a = 0$ and $n = 1$. Describe the graph of the Taylor polynomial $y = T_1(x)$. Find the Taylor polynomial and describe the approximation for $\sin x$ for x close to 0.
2. Set $a = 0$. Consider the graph of the Taylor polynomial $y = T_n(x)$ as n increases. Explain why the graph of the Taylor polynomials for $n = 1$ and $n = 2$ are identical, and for $n = 3$ and $n = 4$, etc.
3. For each value of a and n, describe the accuracy of the Taylor approximation about the point $x = a$.

Page 5.1



This page describes the fourth example, the notation used, and the method for changing the values of n and a. In Problem 5 of this tns file, $f(x) = \cos x$ and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.
Note that in this example, the value of a must be in the interval $[-2\pi, 2\pi]$ and can be changed in increments of $\pi/24$.

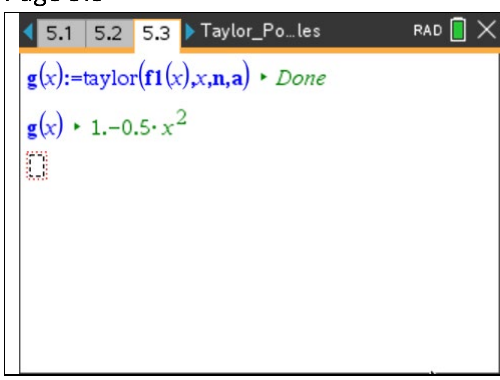
Page 5.2



The graph of $y = f(x) = \cos x$ is black and the graph of $y = T_n(x)$ is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are $n = 1$ and $a = 0$.

Note that it is also possible to grab the point a on the x-axis and move it horizontally to change the value.

Page 5.3



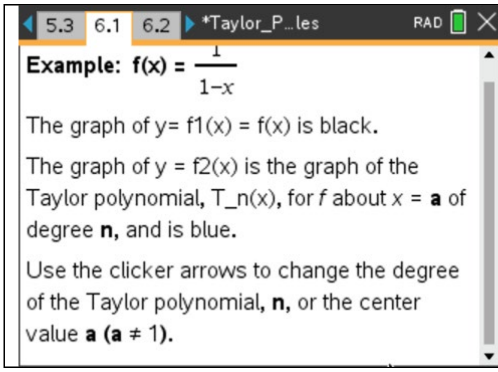
This Notes page defines and displays the Taylor polynomial $g(x) = T_n(x)$ for the current values of a and n.

Problems

Change the values of n and a as necessary and use Pages 5.2 and 5.3 to answer the following questions.

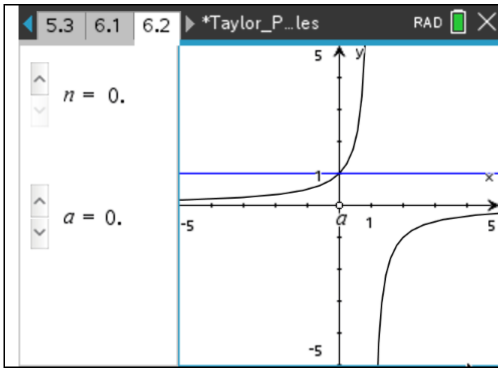
1. Set $a = 0$ and $n = 1$. Describe the graph of the Taylor polynomial. Find the Taylor polynomial and explain why the slope of this linear approximation is 0.
2. Set $a = 0$. Consider the graph of the Taylor polynomial $y = T_n(x)$ as n increases. Explain why the graph of the Taylor polynomials for $n = 0$ and $n = 1$ are identical, and for $n = 2$ and for $n = 3$, etc.

Page 6.1



This page describes the fifth example, the notation used, and the method for changing the values of n and a. In Problem 6 of this tns file, $f(x) = \dfrac{1}{1-x}$ and the graph is presented in black on the next page. The Taylor polynomial is sketched in blue, and the clicker arrows are used to change the degree of the Taylor polynomial and/or the center value.
Note that in this example, $a \neq 1$.

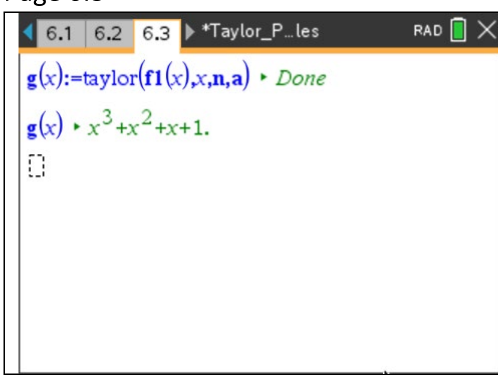
Page 6.2



The graph of $y = f(x) = \dfrac{1}{1-x}$ is black and the graph of $y = T_n(x)$ is in blue. The clicker arrows in the left screen are used to change the degree and/or the center value. The default settings are $n = 0$ and $a = 0$.

Note that it is also possible to grab the point a on the x-axis and move it horizontally to change the value.

Page 6.3



This Notes page defines and displays the Taylor polynomial $g(x) = T_n(x)$ for the current values of a and n.

Problems

Change the values of n and a as necessary and use Pages 6.2 and 6.3 to answer the following questions.

1. Set $a = 0$. Consider the graph of the Taylor polynomial $y = T_n(x)$ for various values of n. Explain why there is no graph of the Taylor polynomial to the right of $x = 1$.

2. Set $a = 0$ and $n = 7$. Explain the accuracy of the Taylor polynomial. Explain the accuracy of the Taylor polynomial $T_7(x)$. Why does the Taylor polynomial appear to be a much better approximation to the right of $x = 0$ than to the left?
3. Explain how to obtain the graph of a Taylor polynomial that can be used to approximate the portion of the graph of $y = \frac{1}{1-x}$ to the right of $x = 1$.

Suggested Extensions

1. Explain how the accuracy of a Taylor polynomial is related to the degree of the Taylor polynomial and the value of a.
2. Describe the interval about $x = a$ on which a Taylor polynomial is fairly accurate.
3. Suppose the function f has a discontinuity at $x = b$. Explain how this value affects a Taylor polynomial for f.
4. What is the relationship between the ith derivative of the function f and the ith derivative of the corresponding Taylor polynomial T_n?
5. Consider exploring the Taylor polynomials associated with the following functions.
   (a) $f(x) = \tan^{-1} x$
   (b) $f(x) = e^{-x^2}$
   (c) $f(x) = e^x \sin x$
   (d) $f(x) = \frac{1}{1+x^2}$
   (e) $f(x) = \tan x$