Math Objectives

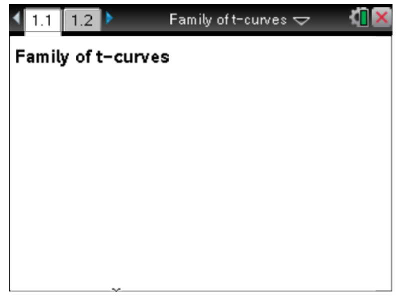
- Students will understand that a family of *t*-distributions is determined by the number of degrees of freedom.
- Students will recognize that a *t*-distribution with a small number of degrees of freedom has more area in the tails than a normal distribution.
- Students will recognize that a *t*-distribution approaches the standard normal distribution as the number of degrees of freedom increases.

Vocabulary

- degrees of freedom
- empirical rule
- mean
- normal probability distribution
- point of inflection
- standard deviation
- *t*-distribution

About the Lesson

- This lesson involves investigating how a *t*-distribution compares to a normal distribution.
- As a result, students will:
  - Compare a *t*-distribution to the standard normal distribution and note that the area in the tails is larger for the *t*-distribution with one degree of freedom.
  - Answer questions about the probability of an outcome occurring for a *t*-distribution with one degree of freedom and compare this probability to that of the same outcome when the distribution is normal.
  - Change the degrees of freedom and observe how the graph of the *t*-distribution changes.
  - Observe that as the degrees of freedom increase, the graph of the *t*-distribution approaches the graph of the standard normal distribution.



TI-Nspire™ Technology Skills:
- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:
- Make sure the font size on your TI-Nspire handhelds is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Files:
Student Activity
Family_of_t-curves_student.pdf
Family_of_t-curves_student.doc
Family of *t*-curves_Create.doc

TI-Nspire document
Family_of_t-curves.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

TI-Nspire™ Navigator™ System

- Use Screen Capture to monitor student progress in creating the tns file.
- Use Quick Poll to check for student understanding.
- Use Live Presenter to demonstrate how to create sliders.

Prerequisite Knowledge

- Students should be familiar with a *t*-distribution and the concept of degrees of freedom as well as with normal distributions and their properties.

Related Lessons

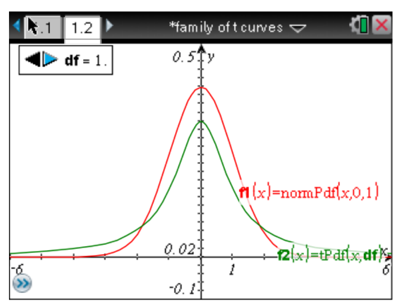
- Statistics Nspired Activity Why *t*?
- Statistics Nspired Activity The Normal Curve Family

Discussion Points and Possible Answers

Tech Tip: Students can create the .tns file following the steps in Family of *t*-curves Create document, or they can use the premade file Family_of_t-curves.tns.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 1 at the end of this lesson.

Move to page 1.2.



1. The graph on Page 1.2 displays the normal probability function with mean 0 and standard deviation of 1. Move the slider to display the graph of a *t*-distribution with one degree of freedom.
   a. Describe the graph.

   <u>Sample Answers:</u> The curve is mound-shaped and symmetric around the vertical axis and is asymptotic to the horizontal axis for large and small *x*-values.

b. How are the graphs of the normal probability distribution and the *t*-distribution alike, and how are they different?

Sample Answers: Both of the curves are mound-shaped and symmetric around the vertical axis (x=0) and are asymptotic to the horizontal axis for large and small x-values. The normal curve has a larger maximum value than the *t*-distribution does when x=0. The *t*-distribution has more area in the tails. The distributions appear to have the same point of inflection, but the normal curve decreases at a faster rate than the *t*-distribution after that point.

2. Select **MENU > Trace > Graph Trace.** Press the right and left arrow keys (the edges of the touchpad) to move the trace point along the curve. The up and down arrow keys on the touchpad will move the point to the other graph.

   a. Estimate the coordinates of the point of inflection for each graph.

   Sample Answers: The standard deviation for the standard normal distribution, which indicates the point of inflection, occurs at x=1. So, the point of inflection for the normal distribution is about (1, 0.242) which corresponds to the *t*-distribution at about (1, 0.159). From the graph, it looks like this might be the point of inflection for the *t*-distribution as well, but it is hard to tell exactly.

   b. Estimate the maximum height of each of the two distributions.

   Sample Answers: The height of the *t*-distribution with one degree of freedom is about 0.318, and the height of the standard normal distribution with standard deviation 1 is about 0.399.

3. Decide whether the following statements are always true, sometimes true, or never true. Explain your reasoning in each case.

   a. The area between the horizontal axis and the normal distribution is smaller than the area between the horizontal axis and the *t*-distribution.

   Sample Answers: Never true, as the area between the horizontal axis and either distribution is always 1 since both are probability distributions.

   b. The probability that an outcome will be greater than 1.8 is larger for the *t*-distribution with one degree of freedom than for the normal distribution.

   Sample Answers: Always true. The tails of the *t*-distribution with one degree of freedom are always above the tails of the normal distribution, and thus the area between the curve and the horizontal axis is greater. The probability P(x > 1.8) is greater for the *t*-distribution with one degree of freedom than for the normal distribution.

c. The probability of an outcome is greater using the *t*-distribution than the normal distribution.

Sample Answers: Sometimes true. Inspecting the graphs suggests that for *x*-values between $\pm$1.8, the area, and thus the probability, is less using the *t*-distribution with one degree of freedom than for the normal distribution. However, as described in b above, the tails of the *t*- distribution with one degree of freedom have larger area and thus a greater probability.

4. Tom claims he can find another normal curve that will exactly match the *t*-distribution with one degree of freedom by changing the mean and standard deviation of the normal distribution.
   a. Do you agree or disagree with Tom? Justify your reasoning.

Sample Answers: If you change the mean, the graph of the normal curve will just be translated horizontally. The shape will remain the same, and the center will no longer match the center of the *t*-distribution. Changing the standard deviation changes the point of inflection and also the height of the normal curve – the curve kind of flattens out for smaller standard deviations or becomes tall for larger standard deviations. But, the percent of area that is more than three standard deviations from the mean will always be very small – less than 0.3% for both tails. If you try matching the standard deviations, the peaks of the two distributions will be different; if you match the peaks, the points of inflection will be different.

   b. Press ctrl **+ G** to bring up the entry line. Click the up arrow twice to see the equation for the normal distribution **f1(x),** which was **normpdf(x,0,1)** where the mean was 0 and the standard deviation was 1. Change the standard deviation to see if your argument for Tom was correct.

Sample Answers: Students should see that even though you can create normal curves (with larger standard deviations) where the areas in the tails nearly match the areas in the tails of the *t*-distribution, the areas – and correspondingly the probabilities – for 1, 2, and 3 standard deviations away from the mean do not match. For example, when the normal pdf has mean 0 and standard deviation 2, the area in the tails might be close to the same for both curves, but the area between -5.5 and -1 is much larger for the normal distribution than for the *t*-distribution with one degree of freedom.

5. Return to the normal distribution with mean 0 and standard deviation 1. A variable of interest related to the *t*-distribution is the number of degrees of freedom.
   a. How do you think the *t*-distribution graph will change if the degrees of freedom, **df**, are increased?

Sample Answers: Student answers could be anywhere from “nothing will change” to “the curve will go higher or flatten out.”

> **Teacher Tip:** Student's *t*-distribution has the probability density function, $f(t)=\dfrac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi}\,\Gamma(\frac{v}{2})}(1+\frac{t^2}{v})^{-.5(v+1)}$, where v is the number of degrees of freedom and Γ is the gamma function, $\Gamma(x)=-\int_0^{+\infty}e^{-i}t^{x-1}dt$. The important thing to note is that the *t*-distribution is not a function of the mean and standard deviation but of the degrees of freedom. This is the point to make with your students and why the *t*-distribution should be used when the population standard deviation is not known. Sharing the equation is not recommended unless the class is **very** strong mathematically.

b. Use the arrow to increase the degrees of freedom, **df**, to 2. Were you right in your conjecture for part a? Explain why or why not.

Sample Answers: I was not right. I thought the new curve would get shorter and more flat. Instead it got skinnier, and the maximum point was higher--closer to the maximum for the normal distribution.

> **Teacher Tip:** Students can use either a minimized slider and click the arrow to increase the degrees of freedom, or they can highlight the region around the arrows and press ctrl **> MENU** > **Maximize** to obtain a slider they can drag to increase the number of degrees of freedom quickly.

c. Use the arrow to increase the number of degrees of freedom, **df**, for the *t*-distribution. Describe what happens to graph as **df** increases.

Sample Answers: The *t*-distribution seems to morph into a normal distribution as the degrees of freedom increase. With 16 degrees of freedom, the curves look almost the same. Using the trace command, when the degrees of freedom are 50, the values for a given input differ by a digit in the thousandths place.

6. Decide whether the following are always true, sometimes true, or never true. Explain your thinking for each answer.
   a. The empirical rule, 68%, 95%, and 99.7%, works for both of the distributions.

   <u>Sample Answers:</u> Sometimes true. The empirical rule is only an approximation to begin with, but the curves are quite different for a small number of degrees of freedom, and so the empirical rule does not work for those *t*-distributions. As the number of degrees of freedom increases, however, the *t*-distributions get closer to the normal distribution, and so the empirical rule will hold for both distributions.

   b. There is really only one *t*-distribution and one normal distribution.

   <u>Sample Answers:</u> Never true. The family of normal distributions depends on the mean and the standard deviation, and the members of the "family" have the same characteristics: mound-shaped and symmetric around the mean, the point of inflection is one standard deviation from the mean, and the empirical rule holds for the area between the curve and horizontal axis. Likewise, the family of *t*-distributions depends on the number of degrees of freedom; the curves are mound-shaped and symmetric around the mean, but the spread changes depending on the degrees of freedom.

   c. The *t*-distribution is a function of the standard deviation and the mean of the population.

   <u>Sample Answers:</u> Never true. The *t*-distribution changes according to the number of degrees of freedom and is not dependent on the mean or the standard deviation of the population.

7. Tinae announced that she is not going to use the normal distribution ever and instead will use the *t*-distribution as it seems like it will give better results overall. What would you say to Tinae?

   <u>Sample Answers:</u> The normal distribution is more accurate in estimating probability of an outcome when the standard deviation of the population from which a sample is drawn is known. If you do not know the standard deviation of the population, it is better to use the *t*-distribution because it is not defined using the standard deviation. Otherwise the results will underestimate the probability in the tails and overestimate the probability near the center. As the number of degrees of freedom increases to above 30 or so, you could use either distribution and have results that are very close.

8. The name of this activity is the Family of *t*-Curves. Does the name make sense? Why or why not?

   Sample Answers: A *t*-distribution exists for each given degree of freedom, so as the degrees of freedom change, you get a whole set of *t*-distributions. This could be thought of as a family of *t*-distributions just as you can get a family of normal distributions by changing the mean and/or standard deviation.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

Wrap Up

Upon completion of the lesson, the teacher should ensure that students are able to understand:

- There is a family of *t*-distributions determined by the number of degrees of freedom
- A *t*-distribution for a small number of degrees of freedom has more area in the tails than a normal distribution.
- As the number of degrees of freedom increases, the *t*-distribution approaches a normal distribution.

Assessment

Choose the most accurate response for each of the questions (correct answers are marked with an *):

1. A *t*-distribution with one degree of freedom is
   a. congruent to a normal distribution.
   b. taller and skinnier than a normal distribution.
   c. has more area in the tails than a normal distribution.*

   Sample Explanation: Looking at the two graphs, it is obvious that the t distribution has more area in the tails than the normal distribution.

2. A z-score of 3 or more is a
   a. common outcome for a normal distribution.
   b. rare outcome for a *t*-distribution with one degree of freedom.
   c. rare outcome for a *t*-distribution with 30 degrees of freedom.*

Sample Explanation: The empirical rule for a normal distribution suggests that the probability of an outcome more than three standard deviations from the mean is less than 0.3%, making the P(X > 3) = 0.15% and P(X<-3) =0.15%. And while the probability of an outcome more than three standard deviations from the mean in a *t*-distribution is not really large, it is larger than for a normal distribution. For a large number of degrees of freedom, the *t*-distribution is close to the normal distribution so something more than three standard deviations from the mean would be rare for both distributions so c is correct.

3. The probability of an outcome less than 0 is
   a. 0.5 for a *t*-distribution*
   b. less than 0.5 for a *t*-distribution
   c. more than 0.5 for a *t*-distribution

Sample Explanation: The mean and the median of the *t*-distribution are 0 for any number of degrees of freedom, so the probability of an outcome less than 0 is 0.5.

4. A *z*-score of -7 or less is a
   a. rare outcome for any *t*-distribution*
   b. likely outcome for a *t*-distribution with one degree of freedom
   c. likely outcome for a normal distribution

Sample Explanation: Inspecting the graphs suggests that an outcome of -7 or less would be in the far tail for both distributions. It would never be likely in a normal distribution and is more unlikely to happen in any *t*-distribution because as the degrees of freedom increase, the tails of the *t*-distribution creep closer to the normal distribution making an extreme outcome less likely. For example, $P(X \leq -7) = 0.045167$ for df = 1, 0.009902 for df = 2, 0.002993 for df = 3.

TI-Nspire Navigator

Note 1

Name of Feature: Live Presenter
Live Presenter may be used in the extension to have a student demonstrate how to build the sliders in the .tns file.

Note 2

Name of Feature: Quick Poll
A Quick Poll can be given at the conclusion of the lesson. The results can be saved and used in a Class Analysis at the start of the next class to discuss possible misunderstandings students might have had.