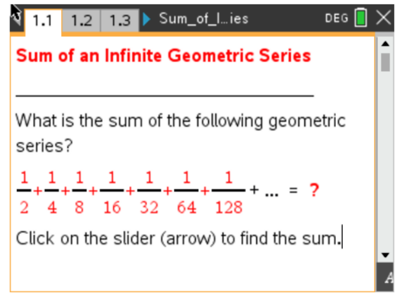


Math Objectives

- Students will understand how a unit square can be divided into an infinite number of pieces.
- Students will understand and justify the sum of an infinite geometric series.
- Students will be able to explain why the sum of an infinite geometric series is a finite number if and only if $|r| < 1$.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- Geometric Series
- Infinite series
- Ratio of a geometric series
- Sigma notation

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations SL/HL and IB Mathematics Approaches and Analysis SL/HL
- This falls under the IB Mathematics Core Content Topic 1 Numbers and Algebra:

  1.3: **(a)** Geometric sequences and series

  (b) Use of the formulae for the n^{th} term and the sum of the 1^{st} n terms of the sequence

  (c) Use of sigma notation for the sums of geometric sequences

  AI HL 1.11: The sum of infinite geometric sequences

  AA 1.9: Sum of infinite convergent geometric sequences

As a result, students will:

- Apply this information to real world situations.

TI-Nspire™ Navigator™

- Transfer a File.
- Use Class Capture to examine patterns that emerge.
- Use Live Presenter to demonstrate.
- Use Teacher Edition computer software to review student documents.
- Use Quick Poll to assess students' understanding

Tech Tips:

- This activity includes screen captures taken from the TI-Nspire CX II handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity

Nspire-SummingUpGeometricSeries-Student.pdf

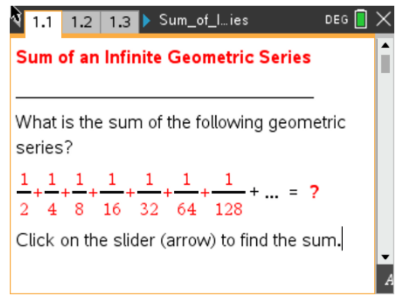
Nspire-SummingUpGeometricSeries-Student.doc

SummingUpGeometricSeries.tns

Activity Materials

Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

In this activity, you will explore the concept of finding the sum of an infinite geometric series. Reviewing the concepts of when you can find the sum of an infinite geometric series will be the first task, discussing with your classmates not only what the characteristics of a geometric sequence are, but also the key characteristic that allows you to add every term of the infinite sequence and still get a non-infinite sum.



Let us review the characteristics of a geometric sequence. A **geometric sequence** is a sequence of terms where the ratio of every two consecutive terms is constant. The constant ratio is referred to as the **common ratio** or ***r***. To find subsequent terms of a geometric sequence, multiply a term by ***r***. The nth term, u_n, formula for a geometric sequence is $\boldsymbol{u_n = u_1 \cdot r^{n-1}}$, where u_1 is the first term, and *r* is the common ratio.

Teacher Tip: Students need to also be aware of the finite geometric series formula $\left(S_n = \frac{u_1(1-r^n)}{1-r}\right)$ and the infinite geometric series formula $\left(S_\infty = \frac{u_1}{1-r}\right)$ that will be discussed in questions throughout the activity.

Problem 1 – Geometric Sequence Practice

1. Find the next three terms of each infinite geometric series.

   (a) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$

   Solution: $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

   (b) $2+\frac{3}{2}+\frac{9}{8}+\cdots$

   Solution: $\frac{27}{32}, \frac{81}{128}, \frac{243}{512}$

2. Discuss with a classmate how you would find the next three terms of each series. Explain your results.

   Solution: First you would find the common ratio by dividing each term by its previous term, then you would multiply this common ratio be the third, fourth and fifth terms.

3. Sigma notation is used at times to express a series. The symbol for sigma, Σ, actually means the sum of. Using the nth term formula from above and the sigma notation, write an expression in terms of *n* that describes each of the series from number 1.

Solution: (a)

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{1}{2}^{n-1}$$

(b)

$$\sum_{n=1}^{\infty} 2 \cdot \frac{3}{4}^{n-1}$$

4. Discuss with a classmate how you would find the sum of each series in number 1. Explain your results.

Possible Discussions: Depending on if you were finding a partial sum or an infinite sum, you could use either of the geometric series formulas. If it is a partial sum, you could simply add the terms. You could use your handheld and sigma notation.

Problem 2 – Finding the Sum of a Geometric Series

There are two types of geometric series. There is the partial sum or finite series and then there is the infinite series.

1. Discuss with a classmate the formula to find a partial sum of a geometric series. Explain what you would need to use the formula and if there are any restrictions.

Solution: Formula: $S_n = \frac{u_1(1-r^n)}{1-r}$

You will need the first term, the common ratio and how many first terms will be added. There could be restrictions if r > 1 and you are finding the sum of all the terms.

2. Explain how you could use sigma notation to find the partial sum of a geometric series as well. Explain what you would need to use sigma.

Solution: You would need to know how many terms you are adding together or which term you are starting with and ending with (x), you would need the common ratio and the first term. This information would be used to substitute in:

$$\sum_{n=1}^{x} u_1 \cdot r^{n-1}$$

3. Discuss with a classmate the formula to find an infinite sum of a geometric series. Explain what you would need to use the formula and if there are any restrictions.

   Solution: Formula: $S_n = \frac{u_1}{1-r}$

   You will need the first term and the common ratio.

   The restriction is if r > 1, then you cannot find the infinite sum, as it would be infinite.

4. Explain how you could use sigma notation to find the infinite sum of a geometric series as well. Explain what you would need to use sigma.

   Solution: You would need to know the first term and the common ratio to substitute in:

$$\sum_{n=1}^{\infty} u_1 \cdot r^{n-1}$$

5. Given the geometric sequence 1, -4, 16, -64, …, find the partial sum of the first 9 terms.

   Solution: $S_9 = 589{,}824$

6. Given the geometric sequence 9, 3, 1, $\frac{1}{3}$, …, find the infinite sum.

   Solution: $S_\infty = 13.5$

7. Write the sequences in questions 5 and 6 in sigma notation. Explain if you can use your handheld to verify your answers using sigma notation.

   Solution: (5)

$$\sum_{n=1}^{9} 1 \cdot (-4)^{n-1} = 589{,}824$$

   (6)

$$\sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{3}\right)^{n-1} = 13.5$$

   Number 5 can be done on either Nspire (CAS or non CAS), number 6 can only be done on the Nspire CAS.

Problem 3 – Visualizing an Infinite Geometric Series

Open the tns file Sum_of_an_Infinite_Geometric_Series.tns. In this file, you will be shown how a unit square can be divided into an infinite number of pieces. **Move to page 1.2.**

1. Select ▲ to see a rectangle with a shaded area of $\frac{1}{2}$ unit^2. The length of a side of the original square is 1 unit. Write down the dimensions of the rectangle.

   Solution: The square measures 1 unit on each side. Since the rectangle contains a side of the square, that side must also measure 1 unit. The area of the rectangle is $\frac{1}{2}$ unit^2. Thus, $lw = 1w = \frac{1}{2}$, so we know that w must equal $\frac{1}{2}$. The dimensions of the rectangle are $1 \times \frac{1}{2}$.

2. Select ▲ until the shaded area increases to $\frac{7}{8}$. Write down the dimensions of the three rectangles whose sum is $\frac{7}{8}$.

Solution: The sum of the areas of the three rectangles is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$

We know from Question 1 that the first rectangle has dimensions $1 \times \frac{1}{2}$.

The second rectangle is formed by halving the side of the first rectangle whose length is 1. Thus, the second rectangle has dimensions $\frac{1}{2} \times \frac{1}{2}$.

The third rectangle is formed by halving the side of the second rectangle whose length is $\frac{1}{2}$. Thus, the second rectangle has dimensions $\frac{1}{4} \times \frac{1}{2}$.

3. Discuss with a classmate what you expect the area of the next rectangle to be added to the sum would be. (Express your area in both fractional and decimal forms.) Explain how you arrived at your conclusion.

   Solution: The next rectangle is formed by halving the side of the third rectangle whose length is $\frac{1}{2}$. Thus, the fourth rectangle has dimensions $\frac{1}{4} \times \frac{1}{4}$. Its area is $\frac{1}{16}$.
   The sum of the areas of the four rectangles is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}=0.9375$

   Teacher Tip: By expressing their answers in decimal form, students can see how close the sum is getting to 1, even after only a few terms are used.

4. Continue selecting ▲ until you can't press it any more. If you could select ▲ again, write down what you think the area of the next rectangle would be. Find what the total sum of the areas would be. (Express your answers in both fractional and decimal forms.)

Solution: The next rectangle is formed by halving a side of the previous rectangle. This would make the area equal to half the area of the preceding rectangle.

Thus, the area of the new rectangle would be $\frac{1}{2} \cdot \frac{1}{1024} = \frac{1}{2048}$.

The area of the shaded region would be $\frac{1023}{1024} + \frac{1}{2048} = \frac{2047}{2048} \approx 0.9995$.

5. If you could continue selecting ▲ an infinite number of times, and the whole region were shaded, describe what the total shaded area would be. (Express your answers in both fractional and decimal forms.)

Solution: Since the square has sides of length 1 unit, we know the area of the square is 1 unit^2.

Thus, the sum of all of the rectangles that would completely shade the region must be 1 unit^2.

6. Write an expression for the sum of the areas of the infinite number of rectangles formed. Find the value of this sum. Explain how you found this sum.

Solution: The sum of the areas of the infinite number of rectangles formed would be:
$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$
Thus, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

7. Express your answer from Question 6 in sigma notation.

Solution: Expressed in sigma notation, we have

$$\sum_{n=1}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} = 1$$

8. Instead of halving the side of the square, suppose that we doubled its size and continued to double a side of each subsequent square formed.
   a. Express the sum of the areas of these squares as an infinite sum.

Solution: If we doubled the side of the square, we would have 2 + 4 + 8 + 16 +…

   b. Describe what happens to this sum as the number of squares increases. Explain your answer.

Solution: If we continued to increase a side of the square, the sum would get infinitely large.

9. Instead of halving the length or width of each of the rectangles, suppose that we multiplied the rectangle's length or width by $\frac{1}{3}$, giving us the series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$

   State if you think that the sum of the series would be finite or infinite. Explain.

   Solution: If we continue to add $\frac{1}{3}$ of the previous term to the sum, the successive terms will continue to get smaller and the sum would approach a finite number. In this case, the sum is $\frac{1}{2}$.

10. Give an example of an infinite geometric series that you think would have a finite sum and an example of one that you think would not have a finite sum. Explain your reasoning.

    Solution: For an infinite geometric series to have a finite sum, the common ratio (r) must be a proper fraction, i.e., $|r| < 1$. Possible examples include 25 + 5 + 1 + 0.2 + … or
    0.1 + 0.01 + 0.001 + 0.0001 + …
    If $|r| \geq 1$, there would not be a finite sum. Possible examples include 1 + 4 + 16 + 64 + …

11. Based on the information above, describe what conjecture that must be true about the ratio of the consecutive terms of an infinite geometric series for the series to have a finite sum.

    Solution: The ratio, r, is conjectured to be a proper fraction, i.e., $|r| < 1$.

Move to page 1.3.

12. Find the values of the ratio r where an infinite geometric series appears to have a finite sum.

    Solution: The ratio r would appear to be a proper fraction, i.e., $|r| < 1$.

<u>Further IB Application</u>

A local coffee shop had an amazing first year after it opened, earning $40,000 of profit. Unfortunately, the profits have been decreasing by 10% each year after the first. Assuming that this trend continues, find the total profits the shop hopes to earn over the course of its lifetime.

Solution: $S_\infty = \frac{40000}{1-0.90} = \$400{,}000$ or

$$\sum_{n=1}^{\infty} 40000 \cdot (0.90)^{n-1} = \$400{,}000$$

Teacher Tip: Throughout this activity, the students are asked to discuss with classmates and explain how they achieved their answers. This is a wonderful opportunity to create a student led classroom. As you float around the room, listen to what they are saying, add to their discussions, and give them leading questions to see how they respond.

TI-Nspire Navigator Opportunity: ***Quick Poll (Open Response)***
Any part to any Problem in the activity would be a great way to quickly assess your student's understanding of finding and discussing both forms of Scientific Notation and Expanded Form.

***Note: This activity has been developed independently by Texas Instruments and aligned with the IB Mathematics curriculum, but is not endorsed by IB™. IB is a registered trademark owned by the International Baccalaureate Organization.*